A PROOF OF A STATEMENT OF BANACH ABOUT THE WEAK* TOPOLOGY

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Let B be a Banach space, and let Γ be a linear manifold in the dual space B^*. Let Γ^1 be the manifold consisting of all the points in B^* that are weak* limits of sequences in Γ. By induction, for every ordinal number ξ we define Γ^ξ as follows (with $\Gamma^0 = \Gamma$):

$$\Gamma^\xi = \Big(\bigcup_{\sigma < \xi} \Gamma^\sigma \Big)^1 .$$

Then $\Gamma \subset \Gamma^1 \subset \Gamma^2 \subset \cdots$, and if ξ has a predecessor, then $\Gamma^\xi = (\Gamma^{\xi-1})^1$. If B is separable, there exists a first countable ordinal ξ_0 such that Γ^{ξ_0} is the weak* closure of Γ; ξ_0 is called the *order of* Γ. Banach, in his discussion of this [1, pp. 208-213], proves that for every positive integer n there exists a linear manifold in ℓ^1 of order n. He then states, but does not prove, that there exist linear manifolds in ℓ^1 of arbitrarily high countable orders. He refers to a paper at this point, but the paper never appeared. The corresponding statement for the space H^∞ has been proved by Sarason [6], [7]. In this paper we shall prove the following.

THEOREM. *If* ξ *is a countable ordinal, there exists an ideal in* ℓ^1 *of order* ξ.

Let c_0 denote the Banach space of all the complex-valued functions on the integer group that vanish at infinity, with the supremum norm. Then $\ell^1 = (c_0)^*$; let $\ell^\infty = (\ell^1)^* = (c_0)^{**}$. Each of the Banach spaces c_0, ℓ^1, ℓ^∞ can be realized as a space of distributions on the circle group (considered as the real numbers modulo 2π), by the correspondence

$$\{\hat{S}(n): -\infty < n < \infty\} \longleftrightarrow \left\{ S(x) \sim \sum_{n=-\infty}^{\infty} \hat{S}(n)\, e^{inx}: 0 \leq x < 2\pi \right\}.$$

Corresponding to c_0, ℓ^1, ℓ^∞, respectively, are the space PF of *pseudofunctions;* the space W of functions with absolutely convergent Fourier series; and the space PM of *pseudomeasures* (see [3, Appendices I to III]).

Under convolution, ℓ^1 is a group algebra; and W, under pointwise multiplication, is its Gel'fand representation. When we refer to a topology in W, we mean the norm topology unless we say otherwise. If I is an ideal (not necessarily closed) in $W \cong \ell^1$, its *hull* is the closed set

$$h(I) = \{x: f(x) = 0 \text{ for every } f \in I\}.$$

The hull h(I) is empty if and only if I = W. If E is a closed set, then the maximal ideal whose hull is E is the closed ideal $I(E) = \{f \in W: f^{-1}(0) \supset E\}$. The minimal ideal whose hull is E is

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$$J(E) = \{f \in W: f^{-1}(0) \text{ is a neighborhood of } E\}.$$

A closed set E is a set of *synthesis* (see [3, Chapter IX]) if the condition $h(I) = E$ determines a unique closed ideal I, or, equivalently, if $I(E)$ equals the closure of $J(E)$.

We shall prove the theorem by constructing a closed set E whose maximal ideal $I(E)$ has the desired order ξ and is furthermore weak*-dense in W, so that $I(E)^\xi = W$. Note that for an arbitrary set E, $I(E)$ is weak*-dense in W if and only if $I(E)^\perp \cap PF = \{0\}$, where $I(E)^\perp$ denotes the annihilator of $I(E)$ in PM.

We need five lemmas. We postpone their proofs to the end.

LEMMA 1. *Let*

$$\eta(E) = \inf\left\{\frac{\limsup_{|n|\to\infty} |\hat{S}(n)|}{\sup_n |\hat{S}(n)|} : S \in I(E)^\perp\right\}.$$

Then $I(E)^1 = W$ *if and only if* $\eta(E)$ *is positive.*

LEMMA 2. *For* $N = 3, 4, \cdots$, *let* E_N *be the closed, perfect set consisting of all the points in* $[0, 1]$ *whose* N*-ary expansion requires no* 1*'s*:

$$E_N = \left\{\sum_{j=1}^{\infty} \varepsilon_j N^{-j} : \varepsilon_j = 0, 2, 3, \cdots, N-2, \text{ or } N-1, \text{ for } j = 1, 2, \cdots\right\}.$$

Then $\eta(E_N) > 0$, *but* $\lim_{N\to\infty} \eta(E_N) = 0$.

LEMMA 3. *If* $x \in E$, $\varepsilon > 0$, *and* ξ *is an ordinal number, then* $x \in h(I(E)^\xi)$ *if and only if* $x \in h[I(E \cap (x - \varepsilon, x + \varepsilon))^\xi]$.

LEMMA 4. *If* $\{F_N\}$ *is a sequence of sets of synthesis such that* $F_N \subset (1/N + 1, 1/N)$, *then the set* $F = \{0\} \cup \bigcup_{N=1}^{\infty} F_N$ *is also a set of synthesis.*

LEMMA 5. *If* I *is an ideal whose hull is a set of synthesis* F, *then* $I^1 = I(F)^1$.

Proof of the theorem. It follows from Lemmas 5 and 1 that every ideal whose hull is a one-point set $\{x\}$ has order 1, since $\{x\}$ is a set of synthesis ([3, Theorem IV, p. 123]) and $\eta(\{x\}) = 1$.

We proceed by induction, considering first the case in which ξ is a limit ordinal. Let $\sigma(n)$ be a one-to-one map of the positive integers onto $\{\sigma: \sigma < \xi\}$. Let $E = \{0\} \cup \bigcup_{n=1}^{\infty} H_n$, where $I(H_n)$ has order $\sigma(n)$ and $H_n \subset (1/(n+1), 1/n)$. Using Lemma 3, we find that $h\left(\bigcup_{\sigma<\xi} I(E)^\sigma\right) = \{0\}$, and thus $I(E)$ has order ξ.

Now consider the case $\xi = 2$. Let $F_N = \{rx + s: x \in E_N\}$, where E_N is the set of Lemma 2, and where r and s are positive reals chosen so that $F_N \subset (1/(N+1), 1/N)$. As the proof of Lemma 2 shows, dilation and translation do not affect the stated properties of E_N. Let $F = \{0\} \cup \bigcup_{N=3}^{\infty} F_N$. Then $\eta(F) = 0$, and therefore, by Lemma 1, $h(I(F)^1) \neq \emptyset$; by Lemmas 1 and 3 and the fact that $\eta(F_N) > 0$ for each N, we see that $h(I(F)^1) = \{0\}$. Hence $I(F)^2 = W$ and $I(F)$ has order 2.

To deal with the case of an ordinal number $\xi > 2$ that has a predecessor, we make use of the set F, which has several useful properties. By a theorem of C. S.

Herz [3, p. 124], the sets F_N are sets of synthesis. Therefore, by Lemma 4, F is a set of synthesis. Finally, F contains a countable dense subset F_0 such that for every $x \in F_0$ there is a nonempty interval $(x, a_x]$ with $(x, a_x] \cap F = \emptyset$. For each $x \in F_0$, let G_x be a set such that $G_x \subset [x, a_x]$, $I(G_x)$ has order $\xi - 1$, and

$$h\Big(\bigcup_{\sigma < \xi - 1} I(G_x)^\sigma\Big) = \{x\}.$$

Let $E = F \cup \bigcup_{x \in F_0} G_x$. Clearly, $h\Big(\bigcup_{\sigma < \xi - 1} I(E)^\sigma\Big)$ contains F, and by Lemma 3 it equals F. By Lemma 5, since F is a set of synthesis, $h(I(E)^{\xi - 1}) = \{0\}$. Therefore I(E) has order ξ. The theorem is proved.

Remark. A set E is a set of *uniqueness* if $J(E)^\perp \cap PF = \{0\}$, or, equivalently, if J(E) is weak*-dense in W. Pyateckiĭ-Šapiro [5, p. 91] mentioned the set F discussed above and pointed out that it is a set of uniqueness with $\eta(F) = 0$. Sections 1 and 3 of his paper [5] prove the remarkable result that there exists a set E that is not a set of uniqueness even though $J(E) \cap PF$ contains no nonzero measures. For an English account of this result, see [4].

It remains to prove the lemmas. Lemma 1 is essentially a remark of Dixmier. To prove it we need the following result.

THEOREM (Banach and Dixmier). *Let* B *be a separable Banach space, and let* Γ *be a weak*-dense linear manifold in* B*. *Let* $j: B \to B^{**}$ *be the canonical identification. Let* $\Gamma^\perp$ *be the annihilator of* Γ *in* B**. *Then* $\Gamma^1 = B^*$ *if and only if the projection* $p_1: jB + \Gamma^\perp \to jB$ *is bounded.*

For a proof of this theorem and related results, see [2, Sections 1 to 6]. We apply it now to the case B = PF, $\Gamma = I(E)$. Lemma 1 will follow when we show that, in fact,

$$\text{(I)} \qquad \|p_1\| = \frac{1 + \eta(E)}{\eta(E)} \quad (\text{possibly} = \infty).$$

Let $\varepsilon > 0$. Then there exists $T \in I(E)^\perp$ and an integer m_0 such that

$$\sup_{|n| > m_0} |\hat{T}(n)| \le (\eta(E) + \varepsilon) \sup_n |\hat{T}(n)|.$$

For $m > m_0$, let $S_m \in PF$ be defined by

$$\hat{S}_m(n) = \begin{cases} -(1 + \eta(E) + \varepsilon)\hat{T}(n) & \text{for } |n| \le m, \\ 0 & \text{for } |n| > m. \end{cases}$$

Then we see that

$$\|p_1\| \ge \frac{\|\hat{S}_m\|_\infty}{\|\hat{S}_m + \hat{T}\|_\infty} \ge \frac{(1 + \eta(E) + \varepsilon)(\sup_{|n| \le m} |\hat{T}(n)|)}{(\eta(E) + \varepsilon)(\sup_n |\hat{T}(n)|)}.$$

Since m is arbitrarily large and ε is arbitrarily small, it follows that

$$\|p_1\| \ge \frac{1 + \eta(E)}{\eta(E)}.$$

By a similar argument, the norm of the projection p_2: $jB + \Gamma^\perp \to \Gamma^\perp$ is at least $1/\eta(E)$; but since clearly

$$S \in PF,\ T \in I(E)^\perp \Rightarrow \frac{\|\hat{T}\|_\infty}{\|\hat{S}+\hat{T}\|_\infty} \le \frac{\|\hat{T}\|_\infty}{\limsup_{|n|\to\infty} |\hat{T}(n)|} \le \frac{1}{\eta(E)},$$

we see that $\|p_2\| = 1/\eta(E)$ and hence that $\|p_1\| \le 1 + 1/\eta(E)$; (I) follows, and Lemma 1 is proved.

Proof of Lemma 2. The quantity $\eta(E_N)$ is positive because E_N is a set of type H (see [3, proof of Theorem III, p. 58]). To show that $\eta(F_N) = O(N^{-1})$, it suffices to show that

$$\text{(II)} \qquad \frac{\limsup_{|t|\to\infty} |\hat{\mu}_N(t)|}{\sup_t |\hat{\mu}_N(t)|} = O(N^{-1}) \quad \text{as } N \to \infty,$$

where μ_N is the *Lebesgue measure on the set* E_N (see [3, pp. 14, 19]), which is the measure supported by E_N and defined as follows. Fix N. For n = 1, 2, ⋯, let λ_n be the measure assigning mass $1/(N-1)$ to each of the $N-1$ points

$$0, \frac{2}{N^n}, \frac{3}{N^n}, \cdots, \frac{N-1}{N^n}.$$

Then $\|\hat{\lambda}_n\|_\infty = \hat{\lambda}_n(0) = 1$ for every n. Let $\mu_{N,n} = \lambda_1 * \lambda_2 * \cdots * \lambda_n$; this measure is supported by the set

$$\left\{ \sum_{j=1}^{n} \varepsilon_j N^{-j} \colon \varepsilon_j = 0, 2, 3, \cdots, \text{ or } N-1 \text{ for } j = 1, \cdots, n \right\}.$$

Let $\mu = \mu_N$ be the weak*-limit of $\{\mu_{N,n}\colon n = 1, 2, \cdots\}$. Then μ is supported by E_N, $\|\hat{\mu}\|_\infty = 1$, and

$$\hat{\mu}(t) = \prod_{k=1}^{\infty} \hat{\lambda}_k(t) = \prod_{k=1}^{\infty} \hat{\lambda}_1(t/N^k) = \hat{\lambda}_1(t)\hat{\mu}(t/N) = \hat{\lambda}_1(t)\hat{\lambda}_2(t)\hat{\mu}(t/N^2)$$

for all real t. Since $|\hat{\mu}(-t)| = |\hat{\mu}(t)| \le |\hat{\mu}(t/N)|$, we can prove (II) by showing that there exists an interval of the form [a, Na], with $a > 0$, on which the quantity

$$|\hat{\lambda}_1(t)\hat{\lambda}_2(t)| = \frac{1}{(N-1)^2} \prod_{k=1}^{2} \left(\frac{1 - e^{-it/N^{k-1}}}{1 - e^{-it/N^k}} - e^{-it/N^k} \right)$$

is bounded by a constant times N^{-1}. We can do this by taking $a = 2\pi/(N+1)$. Lemma 2 is proved.

Proof of Lemma 3. Let g be a function in W that equals 1 at x and vanishes off $(x-\varepsilon, x+\varepsilon)$. It is easy to prove by induction that

$$f \in I(E \cap (x-\varepsilon, x+\varepsilon))^\xi \Rightarrow fg \in I(E)^\xi \quad \text{and} \quad (fg)(x) = f(x).$$

For an arbitrary ideal I, $x \notin h(I)$ if and only if there exists $f \in I$ such that $f(x) \ne 0$. Using these two facts, we can easily prove that

$$x \notin h[I(E \cap (x - \varepsilon, x + \varepsilon))^{\xi}] \Rightarrow x \notin h(I(E)^{\xi}).$$

The converse is obvious. Lemma 3 is proved.

Proof of Lemma 4. We must show that if $f \in I(F)$ and $\varepsilon > 0$, then there exists $g \in J(F)$ such that $\|f - g\|_W < \varepsilon$. For each $\lambda > 0$, we define the function V_λ on $[-\pi, \pi]$ as follows:

$$V_\lambda(x) = \begin{cases} 1 & \text{if } |x| \leq \lambda; \\ 2 - |x|/\lambda & \text{if } \lambda \leq |x| \leq 2\lambda; \\ 0 & \text{if } 2\lambda \leq |x| \leq \pi. \end{cases}$$

Since $f(0) = 0$, we know [3, p. 170] that we may select a small enough $\lambda > 0$ so that $\|f V_\lambda\|_W < \varepsilon/2$. Let $f_0 = f(1 - V_\lambda)$, and let M be an integer large enough so that $f_0(x) = 0$ for $|x| \leq 1/(M+1)$. We may choose $h_1, \cdots, h_M \in W$ so that

$$h_N \in J(F \setminus F_N) \quad \text{for } N = 1, \cdots, M$$

and

$$\sum_{N=1}^{M} h_N(x) = 1 \quad \text{for } 1/(M+1) \leq |x| \leq \pi.$$

Thus $f_0 = \sum_{N=1}^{M} h_N f_0$. Since each F_N is a set of synthesis, there exist functions $f_N \in J(F_N)$ such that

$$\|f_0 - f_N\|_W \leq \frac{\varepsilon}{2M\,\|h_N\|_W}.$$

Let $g = \sum_{N=1}^{M} f_N h_N$. Then $g \in J(F)$ and

$$\|f - g\|_W \leq \|f - f_0\|_W + \left\| \sum_{N=1}^{M} h_N(f_0 - f_N) \right\|_W \leq \varepsilon.$$

Lemma 4 is proved.

Proof of Lemma 5. If $f \in I(F)^1$, there exist $f_n \in I(F)$ such that

$$\text{weak*-}\lim_{n \to \infty} f_n = f.$$

But since F is a set of synthesis, there exist $g_n \in J(F)$ such that

$$\lim_{n \to \infty} \|f_n - g_n\|_W = 0.$$

Therefore

$$\text{weak*-}\lim_{n \to \infty} g_n = f \quad \text{and} \quad f \in J(F)^1 \subset I^1.$$

Therefore $I(F)^1 = I^1$. Lemma 5 is proved.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires.* Chelsea, New York, 1955.

2. J. Dixmier, *Sur un théorème de Banach.* Duke Math. J. 15 (1948), 1057-1071.

3. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques.* Actualités Sci. Indust., No. 1310. Hermann, Paris, 1963.

4. O. C. McGehee, *An account of a result of Pyateckiĭ-Šapiro.* Mimeographed, Dept. of Math., University of California at Berkeley, 1966.

5. I. I. Pyateckiĭ-Šapiro, *Supplement to the work "On the problem of uniqueness of expansion of a function in a trigonometric series"* (in Russian). Moskov. Gos. Univ. Uč. Zap. 165, Mat. 7 (1954), 79-97.

6. D. Sarason, *Weak-star generators of* H^∞. Pacific J. Math. 17 (1966), 519-528.

7. ———, *On the order of a simply connected domain.* Michigan Math. J. 15 (1968), 129-133.

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