DIMENSIONS OF COMPACT TRANSFORMATION GROUPS

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1. INTRODUCTION

A well-known result of Montgomery and Zippin [6], [7] states that the dimension of a compact Lie group of homeomorphisms acting effectively on a connected n-dimensional manifold cannot exceed $n(n + 1)/2$. This result does not hold for actions on nonmanifolds. It is easy to construct effective actions of tori of arbitrarily high dimension on finite connected 2-complexes, and in fact even an effective action of the infinite-dimensional torus on a compact connected 2-dimensional space. For example, consider the 2-complex consisting of a line and r disjoint closed discs with centers on the line. We obtain an effective action of the r-torus T^r on this space by leaving the line point-wise fixed and letting the i^{th} factor group of T^r act as the group of rotations on the i^{th} disc while leaving all other discs pointwise fixed. Clearly, this action has r distinct isotropy subgroups (excluding T^r itself). In Section 2 we consider actions of a compact transformation group G on a space X, and we investigate the connection between the dimension of G and the isotropy structure of the action. The results are simple to state, and the proofs are straightforward.

In [4] it was shown that for an effective action of a compact Lie group H on a connected n-manifold, many dimensions of H less than $n(n + 1)/2$ are also excluded. In Section 3 we show that the same pattern of gaps in dimension occurs for transitive actions of compact non-Lie groups.

Finally, in Section 4 we investigate actions of compact connected non-Lie groups on manifolds. It is of course an unsettled question whether compact non-Lie groups can act effectively on manifolds. By a result of Bredon [2], a compact connected non-Lie group acting on an n-manifold has orbits of dimension at most $n - 3$. Using Bredon's results and the results of Section 3, we show that a compact non-Lie group acting effectively on a connected n-manifold has dimension at most $(n - 4)(n - 3)/2 + 1$.

We assume that all transformation groups are metrizable and that all spaces are separable and metrizable. The reader is referred to [6] or [7] for terms such as *effective*, *free*, *transitive*, *orbit*, *isotropy* or *stability subgroup*, and *orbit space*. The author is grateful to Professor T. S. Wu for his help and encouragement, and in particular for his clever suggestions in the proof of Lemma 1.

2. DIMENSION AND ISOTROPY STRUCTURE

It is known that a compact group G acting transitively and effectively on a finite-dimensional space X is finite-dimensional [6, Theorem 4], [7, p. 239]. In fact, even more is true: $\dim G \leq n(n + 1)/2$, where $n = \dim X$ [6, Theorem 10], [7, p. 243]. However, since there appears to be a slight inaccuracy connected with the proof of Theorem 10 in [6], we prefer to give an alternate proof based on the following theorem. A transformation group G on a space X is said to be *almost effective* if

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there exists a 0-dimensional subgroup G_0 of G acting trivially on X such that the natural action of G/G_0 on X is effective.

THEOREM 1. *Let* G *be a compact group acting transitively and effectively on an* n*-dimensional space* X. *Suppose that* F *is a* 0*-dimensional subgroup such that*

$$G/F = H$$

is a compact Lie group, and let X/F *denote the orbit space of* F *on* X. *Then*

$$\dim X/F = n,$$

and H *is almost effective and transitive on the compact manifold* X/F.

Proof. We show first that $\dim X/F = n$. Let G^* denote the isotropy subgroup of G at x in X. Then

$$X \approx G/G^*. \tag{1}$$

Consider the projection

$$\pi: G \to G/F = H,$$

and let

$$H^* = \pi(G^*).$$

Now H acts transitively on X/F. Letting $\overline{H}^*$ denote the isotropy subgroup at $F(x)$ of this action, we see that

$$X/F \approx H/\overline{H}^*, \tag{2}$$

and we shall show that $H^* = \overline{H}^*$, from which it will follow that

$$n = \dim X = \dim G/G^* = \dim H/H^* = \dim H/\overline{H}^* = \dim X/F. \tag{3}$$

It is fairly obvious that $H^* \subset \overline{H}^*$. We show $\overline{H}^* \subset H^*$. Let

$$h \in \overline{H}^*, \quad h = gF,$$

where $g \in G$, $\pi(g) = h$. Now,

$$F(x) = h[F(x)] = gF[F(x)] = F(gx).$$

Hence, $gx = fx$ for some $f \in F$, and

$$f^{-1}g \in G^*.$$

Finally,

$$h = \pi(g) = \pi(f^{-1}g) \in \pi(G^*) = H^*.$$

Since H is a compact Lie group, it follows from (2) and (3) that X/F is a compact n-manifold.

It remains to show that H is almost effective on X/F. Suppose $K \subset H$ acts trivially on X/F, and let

$$L = \pi^{-1}(K).$$

Now

$$L(x) \subset F(x) \quad \text{for all } x \in X,$$

since $F(x) = \ell F[F(x)] = F(\ell x)$ and therefore $\ell x \in F(x)$ for all ℓ in L. Let L^0 be the identity component of L. Then

$$L^0(x) = x \quad \text{for all } x,$$

since F(x) is 0-dimensional. Therefore L^0 acts trivially on X, and since G was assumed to be effective on X, L^0 is trivial. It follows that

$$\dim K = \dim L = 0.$$

COROLLARY (Montgomery and Zippin). *Let* G *be a compact group acting transitively and effectively on a connected* n-*dimensional space* X. *Then*

$$\dim G \leq n(n+1)/2.$$

Proof. Using Theorem 1, we simply apply the previously mentioned result for compact Lie groups acting effectively on manifolds. Our Theorem 1 replaces the material immediately prior to Theorem 10 of [6].

THEOREM 2. *Let* G *be a compact connected group acting effectively on an* n-*dimensional space* X *with* s *distinct conjugate classes of isotropy subgroups* (G *excluded*). *Then*

$$\dim G \leq sn(n+1)/2.$$

Proof. Let X^* denote the complement of the fixed point set of G. Then G is effective on X^*. Let $X_1, X_2, \cdots, X_s$ denote the point-set unions of the orbits corresponding to the s conjugate classes of isotropy subgroups. Of course,

$$X^* = \bigcup_{j=1}^{s} X_j.$$

We proceed by induction on s. If s = 1, the result actually follows by the Corollary to Theorem 1, since the isotropy subgroups at all points of X_1 are conjugate. Therefore, if T denotes any isotropy subgroup, then

$$K_1 = \bigcap_{g \in G} gTg^{-1}$$

is a normal subgroup of G acting trivially on X_1, and such that G/K_1 is effective on $X_1 = X^*$. Therefore K_1 is trivial. But clearly $G = G/K_1$ is also effective on each single orbit in X_1. The result follows.

For general s, let K_j (j = 1, 2, $\cdots$, s) denote the normal subgroup of G acting trivially on X_j such that G/K_j is effective on X_j. Now

$$G/(K_1 \cap K_2 \cap \cdots \cap K_{s-1})$$

is effective on $X_1 \cup X_2 \cup \cdots \cup X_{s-1}$ with $s - 1$ conjugate classes of isotropy subgroups. Therefore, by the induction hypothesis,

$$\text{(1)} \qquad \dim [G/(K_1 \cap K_2 \cap \cdots \cap K_{s-1})] \leq (s - 1)n(n+1)/2 .$$

But $K_1 \cap \cdots \cap K_{s-1}$ is effective on X_s, and therefore

$$\text{(2)} \qquad \dim (K_1 \cap \cdots \cap K_{s-1}) \leq \dim G/K_s \leq n(n+1)/2 .$$

The result follows from (1) and (2).

To see that Theorem 2 is best possible, consider the effective action of the direct sum of s copies of the special orthogonal group $SO(n + 1)$ on the disjoint union of s n-spheres. If the space X is required to be connected, much sharper results are obtained. We proceed to investigate this situation.

LEMMA 1. *If a compact, finite-dimensional group* G *acts effectively and nontransitively on a connected, locally compact,* n*-dimensional space* X, *then the maximal dimension* r *of the orbits is at most* $n - 1$.

Proof. This result is quite apparent when G and X are both manifolds. If G is a Lie group (X not necessarily a manifold), the result follows from standard local cross-section theorems of Gleason [3]. For choose a point y in X such that the isotropy subgroup G_y has minimal dimension and fewest components. Then $G(y)$ is an orbit of maximal dimension. It follows that there exists an invariant neighborhood W of y such that G_x is conjugate to G_y for all x in W, and consequently the orbits of G have a local cross-section at p [7, pp. 221, 222]. Actually, W is a locally trivial fiber bundle over the orbit space W/G with typical fiber $G(y)$. Since X/G is connected and is not a single point, $\dim X/G > 0$ and each open subset of X/G has positive dimension. Now $G(y)$ is a manifold, and it follows that

$$\dim G(y) < \dim W \leq n .$$

In the general case, we attempt to use a result of Bredon [2] on local cross-sections for actions of compact non-Lie groups. Locally, G is of the form $L \times N$, where L is a local Lie group and N is a compact 0-dimensional group such that G/N is a Lie group. From Theorem 1 it follows that

$$\dim G(x) = \dim G/N[N(x)] \quad \text{for each } x \text{ in } X,$$

where $G/N[N(x)]$ represents the orbit of the action of G/N on X/N at the point $N(x)$. Let W be an invariant set of the type mentioned at the beginning of the proof for the action of G/N on X/N. Then, if π denotes the orbit projection

$$\pi: X \to X/N,$$

the G-orbit of each point of $\pi^{-1}(W)$ has maximal dimension, say k. Let $p \in \pi^{-1}(W)$. By Bredon [2, Theorem 4a], there exist a closed subset C of X containing p, and a closed k-cell K in L such that the natural map

$$K \times C \to K(C)$$

is a homeomorphism onto a compact neighborhood V of p in X contained in $\pi^{-1}(W)$. If we knew that $\dim C > 0$, we would of course be finished. This would always be the case, for example, if X is locally connected. (Since X is assumed to be locally compact, a closed subset has dimension zero if and only if it is totally disconnected.) In any case, at $\pi(p)$ there exists a local cross-section D with respect to the orbits of G/N. Because of the bundle structure of W, we may take D to be a positive-dimensional compact subset of X/N with $D \subseteq \pi(\text{int } V)$. By assuming that C is totally disconnected, we shall arrive at a contradiction.

We first show that $\pi^{-1}(D) \cap V$ is totally disconnected. Now

$$\pi^{-1}(D) \cap V = \pi^{-1}(D) \cap (K \times C),$$

and since C is totally disconnected, each component of $\pi^{-1}(D) \cap V$ is contained in $K \times \{c\}$, for some $c \in C$. It is therefore sufficient to show that each

$$\pi^{-1}(D) \cap (K \times \{c\})$$

is totally disconnected. We claim that

$$\pi^{-1}(D) \cap (K \times \{c\}) = \pi^{-1}(d) \cap (K \times \{c\})$$

for some $d \in D$ (this will yield the desired conclusion, since $\pi^{-1}(d) = Nd$ and N is totally disconnected). Suppose $\pi^{-1}(d_1)$ and $\pi^{-1}(d_2)$ both meet $K \times \{c\}$. Now

$$\pi(K \times \{c\}) = \bigcup_{k \in K} N(kc) \subset G/N[\pi(c)],$$

since $K \cap N = \{e\}$. But D is a local cross-section for G/N on X/N, and therefore $d_1 = d_2$.

Next we show that $\pi^{-1}(D)$ is totally disconnected. We claim that

$$\pi^{-1}(D) = N(V \cap \pi^{-1}(D)).$$

Now $N(V \cap \pi^{-1}(D)) \subset N\pi^{-1}(D) = \pi^{-1}(D)$. Since $D \subseteq \pi(V)$, $\pi^{-1}(D) \subset NV$. Therefore, if $x \in \pi^{-1}(D)$, $x = nv$ for some v in V. But $nv \in \pi^{-1}(D) = N\pi^{-1}(D)$, and therefore $v \in \pi^{-1}(D)$.

Now

$$N(\pi^{-1}(D) \cap V) = \pi^{-1}(D) \subseteq N(\text{int } V)$$

and $\pi^{-1}(D) \cap V$, and hence $N(\pi^{-1}(D) \cap V)$ is compact. Therefore

$$N(\pi^{-1}(D) \cap V) \subseteq n_1 V \cup n_2 V \cup \cdots \cup n_t V$$

for some $n_1, n_2, \cdots, n_t$ in N. But

$$N(\pi^{-1}(D) \cap V) \cap nV = n\{N(\pi^{-1}(D) \cap V) \cap V\} = n(\pi^{-1}(D) \cap V).$$

It has already been observed that $\pi^{-1}(D) \cap V$ is totally disconnected. Therefore $\pi^{-1}(D)$ is a finite union of compact 0-dimensional sets and is therefore 0-dimensional. $\pi^{-1}(D)$ is invariant under N, and since the orbit projection map $\pi_1: \pi^{-1}(D) \to D$ is open, D is also 0-dimensional, which is a contradiction.

It seems likely that both the assumptions that G be finite-dimensional and X be locally compact can be dropped, with a more careful argument.

THEOREM 3. *Let* G *be a compact, connected group acting effectively on a connected, locally compact,* n*-dimensional space* X *with* s *distinct conjugate classes of isotropy subgroups* (G *possibly included*). *Then*

$$\dim G \leq \begin{cases} n(n+1)/2 & (s = 1), \\ (s-1)(n-1)n/2 & (s \geq 2). \end{cases}$$

Proof. The case $s = 1$ follows from Theorem 2. Suppose then that $s \geq 2$. Let the X_j and K_j $(j = 1, 2, \cdots, s)$ be as in the proof of Theorem 2, and let K_j^0 denote the identity component of K_j. Since X is connected, it follows from [6, Theorem 8], [7, p. 242] that $K_i^0 \subset K_k^0$ for some pair i, k $(i \neq k)$. For simplicity, let us suppose $K_2^0 \subset K_1^0 \subset K_1$. Select one orbit $\mathscr{O}_j$ from each X_j for $j \geq 2$, and let

$$\mathscr{O} = \bigcup_{j=2}^{s} \mathscr{O}_j .$$

Since $s \geq 2$, G is not transitive on X, and by Lemma 1 each $\mathscr{O}_j$ (and hence $\mathscr{O}$) has dimension at most $n - 1$.

Now $G/(K_2 \cap \cdots \cap K_s)$ is effective on $\mathscr{O}$ with $s - 1$ conjugate classes of isotropy subgroups. It follows immediately from Theorem 2 that

$$\dim [G/(K_2 \cap \cdots \cap K_s)] \leq (s-1)(n-1)n/2$$

(note that $\mathscr{O}$ is possibly disconnected). To complete the proof, we shall show that

$$\dim (K_2 \cap \cdots \cap K_s) = 0 .$$

Clearly, $\bigcap_{j=1}^{s} K_j = e$. Therefore

$$K_2^0 \cap K_3 \cap \cdots \cap K_s = K_1 \cap K_2^0 \cap K_3 \cap \cdots \cap K_s = e$$

and

$$\dim (K_2 \cap \cdots \cap K_s) = \dim \frac{K_2 \cap K_3 \cap \cdots \cap K_s}{K_2^0 \cap K_3 \cap \cdots \cap K_s} \leq \dim \frac{K_2}{K_2^0} = 0 .$$

To see that Theorem 3 is best possible, consider the effective action (mentioned previously) of the direct sum G of $s - 1$ copies of SO(n) on the disjoint union of $s - 1$ $(n - 1)$-spheres. Let X be the cone over this union, and extend the action of G to X by leaving the vertex of X fixed. Now $\dim X = n$, $\dim G = (s-1)(n-1)n/2$, and there are s conjugate classes of isotropy subgroups, including the isotropy subgroup G at the vertex of X. One might easily run through the proof of Theorem 3 with this example in mind.

The results corresponding to Theorems 2 and 3 when G is abelian are much simpler to state and prove. The examples mentioned in Section 1 are best possible, along these lines. We next turn briefly to actions of locally compact groups, where the situation becomes more complicated. It is possible, for instance, to construct effective actions of the n-dimensional vector group R^n on the plane, for arbitrarily large n. (Compare this with the result of Montgomery and Zippin on effective actions of *compact* Lie groups on manifolds.) We do, however, have the following result.

THEOREM 4. *Let* G *be a locally compact abelian group acting effectively on an* n-*dimensional space* X *with* s *distinct isotropy subgroups* (G *excluded*). *Then*

$$\dim G \leq sn.$$

Proof. We may suppose that G is connected. There exists a closed cell D in G with dim D = dim G. We proceed by induction on s. If s = 1, G acts freely on X and there exists a one-to-one continuous map

$$\alpha: G \to X.$$

If we restrict α to D, we obtain an imbedding, and therefore

$$\dim G = \dim D \leq \dim X = n.$$

In general, let G^* be any isotropy subgroup of G on X, and consider the action of G^* on X. Now G^* acts with at most s - 1 distinct isotropy subgroups (G^* excluded) and therefore, by the induction hypothesis,

$$\dim G^* \leq (s - 1)n.$$

Finally, there exists a continuous one-to-one map of G/G^* onto the orbit corresponding to G^*. By [7, p. 239], G/G^* contains a closed k-cell D_1, where

$$k = \dim G - \dim G^*.$$

It follows that $k \leq n$ and

$$\dim G = k + \dim G^* \leq sn.$$

If we drop the requirement that G be abelian, in Theorem 4, everything falls apart. For example, there exist noncompact Lie groups of arbitrarily high dimension acting transitively and effectively on the plane [8].

3. GAPS IN THE DIMENSIONS OF COMPACT TRANSITIVE TRANSFORMATION GROUPS

The next theorem was proved in [4; see Theorem 2] for the case where G is a compact *Lie* group acting effectively (but possibly not transitively) on an n-dimensional *manifold* X. Let us recall that a compact, connected group acting effectively and *transitively* on an n-dimensional space has dimension at most n(n + 1)/2.

THEOREM 5. *Let* G *be a compact, connected group acting effectively and transitively on an* n-*dimensional space* X. *If the dimension of* G *falls into one of the ranges*

$$\frac{(n-k)(n-k+1)}{2} + \frac{k(k+1)}{2} < \dim G < \frac{(n-k+1)(n-k+2)}{2} \quad (k = 1, 2, \cdots),$$

then there exist only three possibilities:

(i) n = 4, G *is isomorphic to* SU(3)/Z (SU(3) *denotes the special unitary group in complex* 3-*space, and* Z *is its center*), *and* X *is homeomorphic to the complex projective plane* $P^2(C)$.

(ii) $n = 6$, G *is isomorphic to the special group* G_2, *and* X *is homeomorphic to* $P^6(R)$ *or* S^6.

(iii) $n = 10$, G *is isomorphic to* $SU(6)/Z$, *and* X *is homeomorphic to* $P^5(C)$.

Proof. We shall simply reduce the situation to the known result of [4]. Let F be a 0-dimensional subgroup of G such that $G/F = H$ is a compact Lie group. By Theorem 1, H is transitive and almost effective on the compact n-manifold X/F. Therefore, by [4, Theorem 2], $\dim H = \dim G$ cannot fall into any of the ranges above except when

(i) $n = 4$, $\dim G = \dim H = 8$, and H is locally isomorphic to $SU(3)$,

(ii) $n = 6$, $\dim G = \dim H = 14$, and H is isomorphic to G_2,

(iii) $n = 10$, $\dim G = \dim H = 35$, and H is locally isomorphic to $SU(6)$.

It remains to determine the pairs (G, X) for these special cases.

By a theorem of Pontrjagin [9, Example 74, p. 285],

$$G = \frac{A \oplus C}{N}, \tag{1}$$

where

(a) A is a compact, connected, abelian group.

(b) C is a compact, connected, simply-connected, *semisimple Lie* group.

(c) N is a finite normal subgroup.

Using (1), we can easily verify that if $G/F = H$ is semisimple, then A must be trivial and G must also be a semisimple *Lie* group. Since in all the special cases H is simple, G is a compact Lie group and Theorem 2 of [4] settles the question. G is actually determined up to isomorphism type, since we have assumed the action of G to be effective. (See the statement immediately following the proof of Theorem 2 in [4].)

We turn now to an investigation of compact transitive transformation groups of high dimension. By the last theorem, the only permissible dimensions of G greater than $(n-2)(n-1)/2 + 2$, aside from the three isolated examples, are

(A) $n(n+1)/2$,

(B) $(n-1)n/2 + 1$,

(C) $(n-1)n/2$,

(D) $(n-2)(n-1)/2 + 3$.

We wish to obtain a characterization of these cases.

THEOREM 6. *Let* G *be a compact connected group acting transitively and effectively on an* n*-dimensional space* X. *If*

$$\dim G > \frac{(n-2)(n-1)}{2} + 2,$$

one of the following holds.

(i) G *is a Lie group and* X *is a manifold.*

(ii) $\dim G = (n-1)n/2 + 1$, $n \geq 3$, *and* G *is locally isomorphic to* $A^1 \oplus \mathrm{Spin}(n)$, *where* A^1 *is a* 1*-dimensional solenoidal group and* Spin(n) *is the spinor group, the universal covering group of* SO(n).

(iii) $n \leq 3$, G *is an abelian non-Lie* n*-dimensional group, and* (G, X) *is the free action of* G *upon itself by left translation.*

(iv) n = 5 *and* G *is locally isomorphic to* $A^1 \oplus SU(3)$ [*this corresponds to* $\dim G = (n-2)(n-1)/2 + 3$].

Proof. Suppose, as in the proof of Theorem 5, that

$$G = \frac{A \oplus C}{N}. \tag{1}$$

By Theorem 1, there exists a 0-dimensional subgroup F of G such that G/F = H is a compact Lie group acting transitively and almost effectively on the n-manifold M = X/F. It is easy to verify that H has the form

$$H = \frac{T^r \oplus C}{K}, \tag{2}$$

where T^r is an r-torus with r = dim A, where C is the same compact connected, simply-connected, semisimple Lie group of (1), and where K is a finite normal subgroup. We consider the transitive and almost effective action of $\overline{H} = T^r \oplus C$ on the n-manifold M. By [4, Lemma 3], M/T^r is an (n - r)-manifold and C is transitive and almost effective on M/T^r. It follows that

$$\dim C \leq (n-r)(n-r+1)/2 \tag{3}$$

and

$$\dim G = \dim \overline{H} = r + \dim C \leq (n-r)(n-r+1)/2 + r. \tag{4}$$

We use (3) and (4) to investigate the four possibilities for dim G.

(A) dim G = n(n + 1)/2. By (4),

$$n(n+1)/2 \leq (n-r)(n-r+1)/2 + r.$$

The only possibilities are r = 0 and r = n = 1. If 0 = r = dim A, it follows from (1) that G is a compact Lie group. If r = 1 and n = 1, it is easy to see that we have possibility (iii). Since G is now abelian, G acts freely on X, and X is homeomorphic to G. It is now a straightforward exercise to verify that the action is equivalent to left translation.

(B) dim G = (n - 1)n/2 + 1. By (4),

$$(n-1)n/2 \leq (n-r)(n-r+1)/2 + (r-1).$$

Now either $r \leq 1$ or r = 2 and $n \leq 3$. If r = 0, G is a compact Lie group. If r = 1, dim C = (n - 1)n/2 and C is transitive and almost effective on an (n - 1)-manifold. It follows (for example, from [1, Theorem 5]) that C is isomorphic to Spin(n) for $n \geq 3$, and we have possibility (ii). Incidentally, if $n \leq 2$, then $\dim C \leq 1$ and dim C = 0, since C is semisimple. It follows that dim G = dim A = r = 1, n = 1 and we have possibility (iii) again for n = 1. Finally consider r = 2, $n \leq 3$. Since $n - r \leq 1$, it follows from (3) that dim C = 0. Now dim G = dim A = r = 2 and n = 2, and we have possibility (iii)

(C) $\dim G = (n-1)n/2$. By (4),

$$(n-1)n/2 \leq (n-r)(n-r+1)/2 + r .$$

Either $r \leq 1$ or $r = 2$, $n \leq 3$, or $r = 3$, $n = 3$. If $r = 1$, $\dim C = (n-1)n/2 - 1$ and C is transitive and almost effective on an $(n-1)$-manifold. It follows from Theorem 5, if we let $k = 1$ and replace n by $n - 1$, that

$$\dim C = (n-1)n/2 - 1 \leq (n-2)(n-1)/2 + 1 .$$

Consequently, $n \leq 3$ and $\dim C \leq 2$. Therefore $\dim C = 0$, $n = 2$, and $\dim G = 1$, which is impossible since G is transitive on X.

If $r = 2$ and $n \leq 3$, then again $\dim C = 0$ and $(n-1)n/2 = \dim G = \dim A = r = 2$, which is impossible for any integral n. Finally, if $r = 3 = n$, we have possibility (iii).

(D) $\dim G = (n-2)(n-1)/2 + 3$. By (4),

$$(n-2)(n-1)/2 \leq (n-r)(n-r+1)/2 + r - 3 .$$

Now clearly $r \leq 1$. If $r = 1$, then $\dim C = (n-2)(n-1)/2 + 2$ and C is transitive and almost effective on an $(n-1)$-manifold. By Theorem 5, letting $k = 1$ and again replacing n by $n - 1$, we have two choices. Either C is isomorphic to $SU(3)$, which is possibility (iv), or

$$\dim C = (n-2)(n-1)/2 + 2 = (n-1)n/2 .$$

Consequently $n = 3$, $\dim C = 3$, and C is isomorphic to $\mathrm{Spin}(3)$. Again we have possibility (ii) for $n = 3$.

This completes the proof of Theorem 6. It is possible, incidentally, to classify the pairs (G, X) that occur in (i) up to the isomorphism type of G and homeomorphism type of X.

4. COMPACT, CONNECTED NON-LIE GROUPS ON MANIFOLDS

It is known [5] that a compact group G acting effectively on a connected n-manifold must be finite-dimensional. Actually the proof in [5] shows that $\dim G \leq n(n+1)/2$. We prove the following result.

THEOREM 7. *Let* G *be a compact, non-Lie group acting effectively on a connected* n-*manifold* M, *and suppose that* $\dim G > (n-5)(n-4)/2 + 2$. *Then either*

(i) $\dim G = (n-4)(n-3)/2 + 1$ *and* G *is locally isomorphic to* $A^1 \oplus \mathrm{Spin}(n-3)$ $(n \geq 6)$, *or*

(ii) $n = 8$ *and* G *is locally isomorphic to* $A^1 \oplus SU(3)$ [*this corresponds to* $\dim G = (n-5)(n-4)/2 + 3$].

Proof. We may suppose that G is connected. For an orbit R, let K_R^0 denote the identity component of the maximal subgroup K_R of G that acts trivially on R. Choose an orbit N such that G/K_N^0 is of maximal dimension. We shall show that K_N^0 is trivial.

By [6, Theorem 8] and [7, p. 242], there exists an open invariant set U in M such that $N \subset U$ and $K_R^0 \subset K_N^0$ for all orbits R contained in U. Now

$$(1) \qquad G/K_N^0 \approx \frac{G/K_R^0}{K_N^0/K_R^0}.$$

Therefore

$$\dim G/K_R^0 \geq \dim G/K_N^0,$$

and by our assumption on N,

$$\dim G/K_R^0 = \dim G/K_N^0.$$

It follows from (1) that

$$\dim K_N^0/K_R^0 = 0,$$

and since K_N^0/K_R^0 is connected,

$$K_R^0 = K_N^0$$

for all orbits R in U. Therefore the connected group K_N^0 leaves the open n-manifold U pointwise fixed. It follows now from [5, Theorem B] that K_N^0 leaves M pointwise fixed, and therefore K_N^0 must be trivial, and K_N is 0-dimensional.

Finally we consider the effective action of $\overline{G} = G/K_N$ on the orbit N. Since G was assumed to be a non-Lie group, it follows by a result of Bredon [2] that

$$\dim N \leq n - 3.$$

If G/K_N is *not* a Lie group, our result now follows immediately from Theorem 6. If G/K_N is a Lie group and

$$\dim G = (n-3)(n-2)/2 \qquad \text{or} \qquad \dim G = (n-4)(n-3)/2,$$

it is easy to verify, with the help of [4, Lemma 3] and Theorem 5, that G/K_N is semisimple, and hence, that G is a semisimple Lie group, contrary to our hypothesis. This same remark holds, of course, if G/K_N corresponds to any of the three special cases of Theorem 5.

On the other hand, if G/K_N is a Lie group and

$$\dim G = (n-4)(n-3)/2 + 1 \qquad \text{or} \qquad \dim G = (n-5)(n-4)/2 + 3,$$

it is easy to show, by means of [4, Lemma 3] and Theorem 5, that G/K_N is locally isomorphic either to $S^1 \oplus \mathrm{Spin}(n-3)$ $(n \geq 6)$ or to $S^1 \oplus SU(3)$. This completes the proof of Theorem 7. We point out that many special-case considerations for small n were eliminated by our lower-bound assumption on dim G.

Any improvement over Theorem 7 seems to depend entirely on improvements of Bredon's result [2], that is, on tighter bounds on the maximal dimension of orbits for actions of compact non-Lie groups on manifolds. For example, it follows that a solenoidal group cannot act effectively on a 3-manifold. However, if a solenoidal group A^1 acts effectively on a 4-manifold M^4, then the product action

$$A^1 \oplus SO(n-3) \qquad \text{on } M^4 \times S^{n-4}$$

provides an effective action of a compact non-Lie group of dimension

$$(n-4)(n-3)/2+1$$

on an n-manifold for all $n \geq 6$. Similarly,

$$A^1 \oplus SU(3) \quad \text{on } M^4 \times P^2(C)$$

provides such an example for

$$\dim G = (n-5)(n-4)/2+3 \qquad (n=8).$$

The statement of Theorem 7 could be strengthened. In the proof we obtained an effective action of $\overline{G} = G/K_N$ on an orbit N of dimension at most n - 3. Since this action is transitive, the results of Theorem 5 show that many dimensions of $\overline{G}$ (hence of G) less than $(n-5)(n-4)/2+2$ are also impossible.

THEOREM 8. *If* G *is a compact abelian non-Lie group acting effectively on an* n-*manifold* $(n \geq 3)$, *then*

$$\dim G \leq n-3.$$

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