ON TOPOLOGICAL INFINITE DEFICIENCY

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Dedicated to Raymond L. Wilder on his seventieth birthday.

1. INTRODUCTION

Any locally convex, complete, linear metric space is called a Fréchet space. If a Fréchet space is normed, it is a Banach space. The countable infinite product s of lines is an example of a Fréchet space that is not a Banach space. It has recently been shown (see [1], [3], and [5]) that all separable infinite-dimensional Fréchet spaces are homeomorphic to each other. In all of these spaces, as well as in the Hilbert cube I^∞, closed sets of infinite deficiency play an important role. In a Fréchet space X, a closed set K is said to have *infinite deficiency* (or *infinite codimension*) if $X \setminus [K]$ is infinite-dimensional, where [K] denotes the closure of the linear subspace spanned by the elements of K. For the Hilbert cube I^∞, we agree that a closed set K has infinite deficiency if for each of infinitely many different coordinate intervals, K projects onto a single *interior* point of the interval.

As examples of topological theorems dealing with closed sets of infinite deficiency, we mention (i) the result of Klee [7] that in ℓ_2 (the space of square-summable sequences of reals with the norm topology) each homeomorphism between two closed sets of infinite deficiency can be extended to a homeomorphism of ℓ_2 onto itself, and (ii) the result in [2] that if M is a countable union of closed sets of infinite deficiency in s, then $s \setminus M$ is homeomorphic to s. In a sense, a set of infinite deficiency in a space is like a point of the space; but the set itself may be topologically rich, and it may even be homeomorphic to the whole space.

In this paper, we determine what kinds of closed sets have *topological infinite deficiency,* that is, may be carried onto closed sets of infinite deficiency, by some space homeomorphism. Corollary 10.2 gives a characterization of topological infinite deficiency in terms of homotopy properties of the complement of the set. Theorem 10.1 gives similar necessary and sufficient conditions under which a homeomorphism from a closed set onto a closed set of infinite deficiency can be extended to a homeomorphism of the space onto itself. It is worth noting that the same conditions apply to the compact space I^∞ and to each separable, infinite-dimensional Fréchet space.

The methods used in this paper combine refinements of techniques described in [2] with special methods dealing with homotopy properties. Except for Section 9, virtually all the apparatus concerns the Hilbert cube I^∞ and the natural embedding of s as a subset of I^∞. In particular, the apparatus shows (Corollary 10.4) that every homeomorphism of I^∞ onto itself is stable in the sense of Brown and Gluck. This result leads to an affirmative solution of the annulus conjecture for I^∞ (Corollary 10.6).

In Section 11, it is shown that there exists a homeomorphism of I^∞ onto itself carrying the so-called pseudo-boundary into the pseudo-interior.

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In Section 8, we show that for each compact subset M of the pseudo-boundary of I^∞, the union of M and the pseudo-interior is homeomorphic to the pseudo-interior (Theorem 8.5).

2. DEFINITIONS AND NOTATION

Let I^∞ denote the Hilbert cube; we use the representation $I^\infty = \Pi_{j>0} I_j$, where for each j, I_j is the closed interval $[0, 2^{-j}]$. For $(x_j), (y_j) \in I^\infty$, with $x_j, y_j \in I_j$, the metric in I^∞ is given by

$$\rho((x_j), (y_j)) = \sqrt{\sum_{j>0} (x_j - y_j)^2}.$$

We also use ρ to denote the distance function for finite products of I_j's. Let s be defined as $\Pi_{j>0} I_j^\circ$, where for each $j > 0$, I_j° is the open interval $(0, 2^{-j})$. Then $s \subset I^\infty$. We call s the *pseudo-interior*, and $I^\infty \setminus s$, denoted by $B(I^\infty)$, the *pseudo-boundary of* I^∞.

We observe that both s and $B(I^\infty)$ are dense in I^∞. With these conventions, we can regard s as a countable infinite product of open intervals. As such, it is clearly homeomorphic to the countable infinite product of lines, which is the usual representation of s as a topological linear space.

Let Z denote the set of all positive integers. For $\alpha \subset Z$, we write $\alpha' = Z \setminus \alpha$. For any $\alpha \subset Z$, let τ_α denote the projection of I^∞ onto $I_\alpha = \Pi_{j \in \alpha} I_j$. For each $j \in Z$, let τ_j denote the projection of I^∞ onto I_j, and let $\overline{\tau}_j$ denote the projection of I^∞ onto $\Pi_{k \neq j} I_k$. For each $j \in Z$, let

$$W_j(0) = \tau_j^{-1}(0) \quad \text{and} \quad W_j(1) = \tau_j^{-1}(2^{-j}).$$

We call $W_j(0)$ and $W_j(1)$ the *endslices* of I^∞ in the j-direction.

An n-cell $M \subset I^\infty$ is said to be a *standard* n-cell if $M = \Pi_{j>0} M_j$, where M_j is a closed subinterval of I_j° for exactly n indices j, and consists of a single point in I_j° for all other indices. We say that M is a *central standard* n-cell if for each $j \in Z$, $I_j \setminus \tau_j(M)$ consists of two components of equal length.

A *standard cubical decomposition* of a standard n-cell M is a finite collection G of standard n-cells whose union is M, with the property that for all $g, g' \in G$ and all $j \in Z$, either $\tau_j(g) = \tau_j(g')$ or $\tau_j(g) \cap \tau_j(g')$ consists of a single point or is empty.

A subset $R \subset I^\infty$ is called a *closed basic set* if $R = \Pi_{j>0} R_j$, where R_j is a closed subinterval of I_j for each j and coincides with I_j for all but finitely many j. The interior in I^∞ of a closed basic set is called an *open basic set.* A closed basic set R may be regarded canonically as a smaller Hilbert cube R' whose boundary Bd R in I^∞ is a finite union of endslices of R'. Note that B(R') refers to the pseudo-boundary of R', while Bd R is a set-theoretic boundary.

A finite polyhedron $K \subset I^\infty$ is said to be *normal* if $K \subset s$ and $\tau_j(K)$ consists of a single point for all but finitely many $j \in Z$.

Let X denote either I^∞ or any separable, infinite-dimensional Fréchet space. A closed subset K of X is said to have *Property* Z if for each homotopically trivial, nonempty open set U in X, $U \setminus K$ is homotopically trivial and nonempty.

The set $K \subset I^\infty$ is said to be *partially deficient with respect to the set* $\alpha \subset Z$ if for each $i \in \alpha$ the set $\tau_i(K)$ is contained in a closed subinterval in I_i°. If for each $i \in \alpha$ the set $\tau_i(K)$ consists of a single point of I_i°, then K is said to be *deficient with respect to* α. If α is infinite and K is (partially) deficient with respect to α, then K is said to be of *infinite (partial) deficiency*. Observe that if $K \subset s$, these definitions also apply to K as a subset of s.

A homeomorphism h of a space X onto itself is said to be *supported* on $M \subset X$ if h is the identity on $X \setminus M$. We let e denote the identity homeomorphism on any appropriate space, and we let d(h, e) denote the distance of h from the identity. For any $M \subset X$ and any homeomorphism h defined on X, the symbol $h \mid M$ denotes the function h restricted to the set M.

A *β-homeomorphism* h is a homeomorphism of I^∞ onto itself such that $h(s) \supset s$, and a *β^*-homeomorphism* is a β-homeomorphism for which $h(s) = s$.

Corresponding to each homeomorphism f of I^∞ onto itself, we denote by $\alpha(f)$ the set of integers j such that $\tau_j(p) \neq \tau_j(f(p))$ for some point p in I^∞. By $\beta(f)$ we denote the set of integers j such that f is independent of I_j, in other words, such that some homeomorphism f^* of $\Pi_{i \neq j} I_i$ onto itself satisfies the condition

$$f(p, q) = (p, f^*(q))$$

for all $p \in I_j$ and all $q \in \Pi_{i \neq j} I_i$.

A sequence $(g_i)_{i>0}$ of homeomorphisms of a compact metric space X onto itself is said to *converge* to a homeomorphism g of X onto itself provided for each $\varepsilon > 0$ there corresponds an integer N such that $d(g^{-1} g_n, e) < \varepsilon$ whenever $n > N$.

For any sequence $(f_i)_{i>0}$ of homeomorphisms of a compact metric space X onto itself for which the sequence $(f_i \circ f_{i-1} \circ \cdots \circ f_1)_{i>0}$ converges to a homeomorphism f of X onto itself, we call f the *infinite left product* of $(f_i)_{i>0}$, and we write $f = L\Pi_{i>0} f_i$ (or $f = L\Pi f_i$).

The following lemma gives a criterion for the existence of the infinite left product of a sequence $(f_i)_{i>0}$ of homeomorphisms of a compact metric space onto itself. The criterion is known, and it has been used occasionally; we underscore it because it seems to be particularly valuable in infinite-dimensional product spaces. Because it involves the distance of f_i from the identity, it may appropriately be used in the inductive definition of a sequence $(f_i)_{i>0}$ of homeomorphisms to ensure existence of the infinite left product.

Notation. For any $\varepsilon > 0$ and any homeomorphism h of a metric space onto itself, let

$$A(h, \varepsilon) = \{\theta \mid \text{for some two points } x, y \in X,\ d(x, y) > \varepsilon \text{ and } d(h(x), h(y)) = \theta\}.$$

Let $a(h, \varepsilon) = \text{g.l.b. } A(h, \varepsilon)$. If X is compact, then it follows from the uniform continuity of h^{-1} that $a(h, \varepsilon) > 0$. Also, in cases where X is compact, $a(h, \varepsilon)$ is sometimes called the *modulus of continuity* of h^{-1}.

LEMMA 2.1. *If* $(f_i)_{i>0}$ *is a sequence of homeomorphisms of a compact metric space* X *onto itself and if*

$$d(f_i, e) < \min((3^{-i}), (3^{-i}) \cdot a(f_{i-1} \circ \cdots \circ f_1, 2^{-i}))$$

for each $i > 1$, *then* $L\Pi_{i>0} f_i$ *exists.*

Proof. Clearly, since $d(f_i, e) < 3^{-i}$, the Cauchy criterion implies that for any $p \in X$, $(f_i \circ \cdots \circ f_1(p))_{i>0}$ converges to a point p' of X. Thus, if we define $f(p) = p'$, then f is a function from X into X. But since for each i the function f_i is continuous and onto, and $d(f_i, e) < 3^{-i}$, we also know that f is continuous and onto. Finally, f is one-to-one, since for each $\varepsilon > 0$ no two points at distance greater than ε can have the same image under f.

3. STABILITY OF HOMEOMORPHISMS OF I^∞

In this section we lay the foundation for the later proof (Corollary 10.4) that every homeomorphism of I^∞ onto itself is stable in the sense of Brown and Gluck [4].

Definition. A homeomorphism of a space X onto itself is *stable* if it is a finite product of homeomorphisms each of which is the identity on some open set.

In [2; see p. 201] we remarked that the homeomorphisms introduced in [2] could all be specified to be stable. Here we shall amplify this statement, since we wish to use the stability of some of these homeomorphisms. In his dissertation, Raymond Wong has given an elementary proof, based on results proved in [1] and [2], that every homeomorphism of s onto itself (or of ℓ_2 onto itself) is stable. He also has shown independently that every homeomorphism of ℓ_2, s, or I^∞ onto itself is isotopic to the identity. Incidentally, we obtain the theorems of this section without using Wong's isotopy theorem.

We note first the trivial facts that every finite product of stable homeomorphisms is stable and that any inverse of a stable homeomorphism is stable.

We now establish an essential elementary lemma, which is a special case of Theorem 3.7. We shall give a proof whose applicability to this lemma was suggested to me by Raymond Wong. He uses a similar argument in his dissertation.

LEMMA 3.1. *Let* h *be a homeomorphism of* I^∞ *onto itself, and suppose there exists an endslice* W *such that* $h(W) = W$ *and* $h \mid W$ *is isotopic to the identity. Then* h *is stable.*

Proof. Without loss of generality, we may consider W to be an endslice in the i-direction, with $W = W_i(0)$. Let J^∞ be $\Pi_{j>0} X_j$, where $X_j = I_j$ for $j \neq i$, and where $X_i = [-2^{-i}, 2^{-i}]$. Then I^∞ is a closed half-space of J^∞, with boundary W. For each t $(0 \leq t \leq 1)$, let V_t denote the level homeomorphism of an isotopy from $V_0 = h \mid W$ to $V_1 = e \mid W$. For each t $(0 \leq t \leq 1)$, let Y_t be the set of all points of J^∞ for which the i^{th} coordinate is $-t/2^{i+1}$. Let $\tilde{h}$ be the homeomorphism of J^∞ onto itself that coincides with h on I^∞, with e on each Y_t for $1 \leq t \leq 2$, and with V_t for each t $(0 < t < 1)$, with Y_t identified canonically as W. Let U be the $2^{-(i+1)}$-neighborhood of $W_i(1)$ in J^∞. Let μ_i be a homeomorphism of $[-2^{-i}, 2^i] = X_i$ onto itself such that μ_i is the identity on $\tau_i(U \cup \tilde{h}^{-1}(U))$ and carries $\tau_i(W)$ onto $-2^{-i} + 2^{-(i+2)}$. Let μ be the homeomorphism of J^∞ onto itself defined coordinatewise as μ_i on X_i and as the identity on X_j for all $j \neq i$.

Clearly, $(\mu^{-1}\tilde{h}^{-1}\mu)$ is the identity on some neighborhood of W, since $\tilde{h}^{-1}$ is the identity on a neighborhood of $\mu(W)$. Thus $(\mu^{-1}\tilde{h}^{-1}\mu)$ carries I^∞ onto itself. Also, by the definition of μ, $\tilde{h} \circ (\mu^{-1}\tilde{h}^{-1}\mu)$ is the identity on U and carries I^∞ onto itself. We define h^* as $\tilde{h}(\mu^{-1}\tilde{h}^{-1}\mu)$, and we note that $\tilde{h} = h^*(\mu^{-1}\tilde{h}^{-1}\mu)^{-1}$. Since each of h^* and $(\mu^{-1}\tilde{h}^{-1}\mu)$ carries I^∞ onto itself, we see that

$$h = \tilde{h} \mid I^\infty = (h^* \mid I^\infty) \circ [(\mu^{-1}\tilde{h}^{-1}\mu) \mid I^\infty],$$

and therefore h is the product of two homeomorphisms each of which is the identity on some open set. Thus h is stable.

Since the identity homeomorphism is automatically isotopic to the identity, we also have the following result.

COROLLARY 3.2. *Let* h *be a homeomorphism of* I^∞ *onto itself such that* $h \mid W$ *is the identity on* W *for some endslice* W *of* I^∞. *Then* h *is stable.*

Now we are in a position to justify the additional conclusion of stability in those theorems of [2] that we shall use below.

THEOREM 3.3. *For each compact set* $K \subset s$ *there exists a stable* β^**-homeomorphism* h *such that* h(K) *has infinite deficiency.*

This theorem (without the condition of stability) is given as Lemma 3.3 of [2]. Using the methods of the proof given in [2], we may require that h is the identity in some neighborhood U of an endslice for which $K \cap \mathrm{Cl}\, U = \emptyset$. By definition, such an h is stable.

In the proof of Theorem 8.1 we shall need the following corollary of Theorem 3.3. It is not explicitly given in [2].

COROLLARY 3.4. *Let* K *be a compact set in* I^∞, *and let* α *be an infinite subset of* Z *such that* $\tau_i(K) \subset I_i^0$ *for each* $i \in \alpha$. *Then there exists a stable* β^**-homeomorphism* g *such that* g(K) *is deficient with respect to some infinite subset* α_1 *of* α.

Proof. Let K_α be the projection of K on $I_\alpha = \Pi_{j \in \alpha} I_j$. Then, regarding I_α canonically as a Hilbert cube, we see by Theorem 3.3 that for some stable β^*-homeomorphism h of I_α onto itself, $h(K_\alpha)$ is deficient with respect to some infinite subset α_1 of α. But then, letting $I^\infty = I_\alpha \times (\Pi_{j \notin \alpha} I_j)$, we can define the desired homeomorphism g coordinatewise as (h, e), with h acting on I_α and e acting on $\Pi_{j \notin \alpha} I_j$.

THEOREM 3.5. *Each homeomorphism* f *of a compact subset* K *of* s *into* s *can be extended to a stable* β^**-homeomorphism* F *of* I^∞ *onto itself.*

In [2], the proof of this theorem (without the explicit use of stability) involves the following three steps.

(1) Exhibiting a β^*-homeomorphism g of I^∞ onto itself such that both g(K) and g(f(K)) have infinite deficiency. Since $K \cup f(K)$ is compact, our Theorem 3.3 implies that we may require g to be stable.

(2) Exhibiting a β^*-homeomorphism h that carries g(f(K)) into a set of infinite deficiency which is complementary to a set of infinite deficiency containing g(K). The homeomorphism h can be achieved as the product of two β^*-homeomorphisms h_1 and h_2, where h_1 involves interchanging just two coordinates (and is thus stable), and where h_2 involves interchanging infinitely many pairs of coordinates, but carries one endslice W onto itself. But such an h_2 is clearly isotopic to the identity under an isotopy carrying W onto itself, and Lemma 3.1 implies that h_2 is stable. Thus we can require that h is stable.

(3) Exhibiting a β^*-homeomorphism ϕ that extends $(hgfg^{-1})$ (with the indicated homeomorphisms cut down to the appropriate sets) from g(K) to hgf(K). The argument for the existence of ϕ is essentially that of Klee [7], and since $g(K) \cup hgf(K)$ is compact, we may clearly require that ϕ is the identity in some neighborhood of an endslice. Thus we may require ϕ to be stable.

Finally we observe that we may define $F = g^{-1} h^{-1} \phi g$ to be the desired stable homeomorphism.

THEOREM 3.6. *For each endslice* W *of* I^{∞}, *there exists a stable* β*-homeomorphism* θ *that carries some compact subset* K *of* s *onto* W.

The proof of this theorem is essentially accomplished in [2; Theorems 6.1 and 5.7], and it is possible to require that θ is the identity in a neighborhood of the opposite endslice in I^{∞}. Thus θ can be required to be stable.

We now give the main theorem of this section.

THEOREM 3.7. *If a homeomorphism* ρ *of* I^{∞} *onto itself either*

(i) *carries an endslice onto itself or*

(ii) *carries an endslice into* s,

it is stable.

Proof. *Case* (i). Suppose ρ carries the endslice W onto W. Let η be a stable homeomorphism carrying W into s, and let μ be a stable homeomorphism extending $(\eta\rho\eta^{-1}) \mid \eta(W)$. Then μ carries $\eta(W)$ onto itself, and $(\eta^{-1}\mu^{-1}\eta)\rho$ is the identity on W and is therefore stable, by Corollary 3.2. Hence $\rho = (\eta^{-1}\mu\eta)[(\eta^{-1}\mu^{-1}\eta)\rho]$ is the product of two stable homeomorphisms, and thus it is stable.

Case (ii). Suppose ρ carries W into s. By Theorem 3.6, some stable homeomorphism σ carries some copy W' of W onto W (with $W' \subset s$). By Theorem 3.5, some stable homeomorphism f carries $\rho(W)$ onto W'. The homeomorphism $\sigma f \rho$ carries W onto W, and by Case (i) it is stable. Therefore $\rho = f^{-1}\sigma^{-1}(\sigma f \rho)$ is stable, since it is the product of three stable homeomorphisms.

The following obvious corollary of Theorem 3.7 justifies the statement that all the homeomorphisms introduced in [2] are stable. Of course, our later Corollary 10.4 asserts even more, namely that *all* homeomorphisms of I^{∞} onto itself are stable.

COROLLARY 3.8. *Any finite product* $f_k \circ \cdots \circ f_1$ *of homeomorphisms of* I^{∞} *onto itself is stable provided* $\alpha(f_i) \neq Z$ *for each* i $(1 \leq i \leq k)$.

4. EXISTENCE LEMMAS FOR HOMEOMORPHISMS

In this section we give a sequence of three lemmas, the third of which is used explicitly in the proof of Theorem 8.1. The lemmas are quite similar in spirit to several of the lemmas and theorems of Section 5 of [2]. We shall prove Lemmas 4.2 and 4.3 in detail, but we shall discuss the proof of Lemma 4.1 only briefly, since it is essentially included in [2].

LEMMA 4.1. *Let* $\varepsilon > 0$, *let* $\bar{\alpha}$ *be an infinite subset of* Z, *and let* K *be a closed subset of an endslice* $W = W_i(0), W_i(1)$ *such that* K *is deficient with respect to* $\bar{\alpha}$. *Then there exist a stable* β*-homeomorphism* h *and a closed set* Q *in the* ε*-neighborhood* $S_\varepsilon(W)$ *of* W *such that*

(1) $\alpha(h) \subset (\bar{\alpha} \cup \{i\})$ *and* $\alpha(h) \neq Z$,

(2) $d(h, e) < \varepsilon$ *and* h *is supported on* $S_\varepsilon(W)$,

(3) $\bar{\tau}_i(Q) = \bar{\tau}_i(K)$ *and* $\tau_i(Q)$ *consists of a single point of* I_i^o,

(4) $h(Q) = K$,

(5) $h(W) \subset W$ *and* $h^{-1}(W) \subset W \cup Q \cup \left[\bigcup_{j \in \bar{\alpha}} [W_j(0) \cup W_j(1)] \right]$, *and*

(6) *for each* $j \in \bar{\alpha}$ *and* $\lambda = 0, 1$, $h(W_j(\lambda)) \subset [W \cup W_j(\lambda)]$ *and* $h^{-1}(W_j(\lambda)) \subset W_j(\lambda)$.

Proof. The proof of this lemma is essentially given as the proof of Lemma 5.1 in [2], with the ε-condition implicit. It involves a type of infinite twisting operation by use of various different $\bar{\alpha}$-directions with motion of Q toward W. Thus we may develop h as a left product $L\Pi_{j>0} h_j$, where for each j, $\alpha(h_j) = \{i, \lambda_j\}$ with $\lambda_j \in \bar{\alpha}$, and where $\alpha(h) \neq Z$. It follows from Corollary 3.8 that h is stable.

LEMMA 4.2. *Let* α *be an infinite subset of* Z, *and let* R *be a subset of* $B(I^\infty)$ *such that* R *is closed relative to* $B(I^\infty)$ *and* R *is deficient with respect to* α. *Then there exist an infinite subset* α_0 *of* α *and a stable* β*-homeomorphism* f *such that* $f(s) = s \cup R$ *and* $\alpha(f) \cap \alpha_0 = \emptyset$.

Proof. Let $(\lambda_i)_{i>0}$ be a monotonic increasing sequence of all indices λ for which $R \cap [W_\lambda(0) \cup W_\lambda(1)] \neq \emptyset$. Then $\alpha \cap \{\lambda_i\}_{i>0} = \emptyset$. Let

$$R_{\lambda_i} = R \cap [W_{\lambda_i}(0) \cup W_{\lambda_i}(1)].$$

Then for each $i > 0$, R_{λ_i} is closed and deficient with respect to α. Let $\alpha = \bigcup_{i\geq 0} \alpha_i$, where α_i is infinite for each $i \geq 0$ and where $\alpha_i \cap \alpha_j = \emptyset$ whenever $i \neq j$. We inductively define a sequence $(f_i)_{i>0}$ of homeomorphisms such that $f = (L\Pi_{i>0} f_i^{-1})^{-1}$. The f_i are to be chosen inductively in such a way that $(f_i^{-1})_{i>0}$ satisfies the convergence criterion of Lemma 2.1. For each $i > 0$, let

$$R_{\lambda_i}(0) = R_{\lambda_i} \cap W(0) \quad \text{and} \quad R_{\lambda_i}(1) = R_{\lambda_i} \cap W_{\lambda_i}(1).$$

For each i, let $f_{i,0}$ and $f_{i,1}$ be β-homeomorphisms of the type described in Lemma 4.1, supported on disjoint neighborhoods of $W_{\lambda_i}(0)$ and $W_{\lambda_i}(1)$, respectively. Here λ_i plays the role of i, and α_i plays the role of $\bar{\alpha}$. For $i = 1$, $R_{\lambda_i}(0)$ and $R_{\lambda_i}(1)$ separately play the role of K, and for $i > 1$,

$$f_{i-1}^{-1} \circ \cdots \circ f_1^{-1}(R_{\lambda_i}(0)) \quad \text{and} \quad f_{i-1}^{-1} \circ \cdots \circ f_1^{-1}(R_{\lambda_i}(1))$$

separately play the role of K. Also, $f_{i,0}$ [or $f_{i,1}$] is the identity if and only if $R_{\lambda_i}(0)$ [or $R_{\lambda_i}(1)$] is empty. We left $f_i = f_{i,0} \circ f_{i,1}$, and we shall now verify that $(L\Pi f_i^{-1})^{-1}$ is the desired homeomorphism f.

(1) For each point p in R, $f^{-1}(p) \in s$, since $L\Pi f_i^{-1}$ moves p off every endslice on which it appears and does not move p onto any endslice on which it does not appear.

(2) $f^{-1}(s) \subset s$, since $f_i^{-1}(s) \subset s$ for each i, and since for each endslice W there is at most one index j for which f_j^{-1} moves any point toward W.

(3) Each point $p \in B(I^\infty) \setminus R$ occurs on some endslice, and if for some j $f_j^{-1} \circ \cdots \circ f_1^{-1}(p)$ is not on an endslice, then $f_j^{-1} \circ \cdots \circ f_1^{-1}(p)$ is on some new endslice with index in $(\alpha_1 \cup \cdots \cup \alpha_j)$ from which it is not moved by any f_k for $k > j$.

(4) Since $\alpha(f) \neq Z$, the homeomorphism f is stable (Corollary 3.8).

LEMMA 4.3. *Let* α *be an infinite subset of* Z. *Let* h *be a* β*-homeomorphism such that* $Cl[h(s) \cap B(I^\infty)] \cap B(I^\infty)$ *is deficient with respect to* α. *Then there exists a stable homeomorphism* g *such that* (gh) *is a* β^**-homeomorphism and* $\alpha(g) \cap \alpha_1 = \emptyset$ *for some infinite subset* α_1 *of* α.

Proof. Let $R = Cl[h(s) \cap B(I^\infty)] \cap B(I^\infty)$. We may write $R = \bigcup_{j \geq 1} R_j$, where for each j, R_j is the nonempty intersection of R with an endslice. Thus R_j is a closed set. By Lemma 4.2, there exists a β-homeomorphism f such that $f(s) = s \cup R$ and $\alpha(f) \cap \alpha_2 = \emptyset$ for some infinite subset α_2 of α. Observe that $(f^{-1} h)$ carries s into s, but that it may also carry some points of $B(I^\infty) \setminus h(s)$ into s. We now find a homeomorphism ϕ that carries such points off s.

Since $I^\infty \setminus s$ is a countable union of compact sets, so is $I^\infty \setminus h(s)$. Therefore, for each j, $R_j \setminus h(s)$ is the countable union of compact sets, and thus $R \setminus h(s)$ may be expressed as $\bigcup_{i>0} Q_i$, where for each i, Q_i is a compact set such that $f^{-1}(Q_i)$ is infinitely deficient with respect to α_2. By Lemma 5.2 of [2], there is an infinite subset α_1 of α_2 and a β-homeomorphism ϕ with $\alpha(\phi) \cap \alpha_1 = \emptyset$ such that for $p \in s$, $\phi(p) \in B(I^\infty)$ if and only if $p \in \bigcup_{i>0} f^{-1}(Q_i)$.

Thus (ϕf^{-1}) is the desired homeomorphism g of the lemma, since $(\phi f^{-1} h)$ carries s onto s and $\alpha(\phi f^{-1}) \cap \alpha_1 = \emptyset$. As in the preceeding lemmas, since $\alpha(g) \neq Z$, g is stable, by Corollary 3.8.

5. TWO PRELIMINARY LEMMAS ON EXTENDING HOMEOMORPHISMS

In proving the key lemma of Section 6, we shall need the following two lemmas, which will be shown to follow from results of [2].

LEMMA 5.1. *Let* R *be a closed basic set, and let* K_1 *and* K_2 *be normal finite polyhedrons in* I^∞ *such that*

$$K_1 \cup K_2 \subset R \quad \textit{and} \quad K_1 \cap \text{Bd } R = K_2 \cap \text{Bd } R.$$

Let f *be a homeomorphism of* K_1 *onto* K_2 *such that* f *is the identity on* $K_1 \cap \text{Bd } R$. *Then there exists a stable* β^**-homeomorphism* g *such that*

$$g \mid K_1 = f \quad \textit{and} \quad g \mid (I^\infty \setminus R) = e \mid (I^\infty \setminus R).$$

Proof. Except for the β^*-condition on g, this lemma is a corollary of Theorem 7.1 of [2] (with stability following by definition). However, to get the full lemma, we must use several results of [2] in sequence. We consider R as a "smaller-scale" Hilbert cube R' with pseudo-interior s'. Since Bd R in I^∞ is the union of a finite number of endslices of R', some β^*-homeomorphism θ of R' carries Bd R into one endslice W of R' (let θ be induced by a homeomorphism of an n-cube onto itself). By Theorem 6.1 of [2], some β^*-homeomorphism ρ of R' carries $\theta(\text{Bd } R)$ into the relative pseudo-interior of W. By applying Lemma 5.1 of [2], we see that there exists a β-homeomorphism ϕ such that $\phi(s') = s' \cup \rho\theta(\text{Bd } R)$. Now the sets

$$K_1' = [\phi^{-1} \rho\, \theta(K_1 \cup \text{Bd } R)] \quad \text{and} \quad K_2' = [\phi^{-1} \rho\, \theta(K_2 \cup \text{Bd } R)]$$

are compact subsets of s'. Let f_0 be the homeomorphism of $(\text{Bd } R) \cup K_1$ onto $(\text{Bd } R) \cup K_2$ that reduces to f on K_1 and to the identity on Bd R. Then (with the indicated homeomorphisms appropriately restricted) $(\phi^{-1} \rho\, \theta) f_0 (\phi^{-1} \rho\, \theta)^{-1}$ is a homeomorphism of K_1' onto K_2'. By Theorem 4.2 of [2], some β^*-homeomorphism σ of R' onto itself extends $(\phi^{-1} \rho\, \theta) f_0 (\phi^{-1} \rho\, \theta)^{-1}$. Thus $(\phi^{-1} \rho\, \theta)^{-1} \sigma (\phi^{-1} \rho\, \theta)$ is a β^*-homeomorphism of R' onto itself that extends f_0. We observe that $(\phi^{-1} \rho\, \theta)^{-1} \sigma (\phi^{-1} \rho\, \theta)$ is a β^*-homeomorphism, since it carries Bd R identically onto itself and each of the seven indicated homeomorphisms carries every other point of $B(R')$ into $B(R')$ and every point of s' into s'. Let σ_0 be the extension of $(\phi^{-1}\rho\,\theta)^{-1}\sigma(\phi^{-1}\rho\,\theta)$ to I^∞ such that for each $p \in I^\infty \setminus R$, $\sigma_0(p) = p$. Then σ_0 is the desired homeomorphism.

LEMMA 5.2. *Let* $\varepsilon > 0$, *and let* R *be a closed basic set in* I^∞. *Let* T *be a normal finite polyhedron in* R, *let* K' *be a closed subset of* R, *and let* g *be a map of a closed* n*-cell* E^n *into* $(\text{Int } R)\setminus K'$ *such that* $g \mid B(E^n)$ *is a piecewise linear homeomorphism onto a polyhedral subset of* T. *Then there exists a homeomorphism* g^* *of* E^n *into* Int R *such that*

(1) $g^* \mid B(E^n) = g \mid B(E^n)$,

(2) $g^*(E^n)$ *is a normal finite polyhedron,*

(3) $g^*(E^n) \cap (K' \cup T) = g(B(E^n))$,

(4) $d(g, g^*) < \varepsilon$.

Proof. Let U be a neighborhood of $g(E^n)$ such that $U \subset (\text{Int } R)\setminus K'$. For each point $p \in E^n$, let V_p be a spherical neighborhood of p in E^n such that $g(V_p)$ is contained in some open basic set that has diameter less than $\varepsilon/3$ and is a subset of U.

Let V^* be the collection of all such sets V_p, and let V' be a finite subcollection of V^* covering E^n. Let Δ be a triangulation of E^n compatible with g and so fine that for each n-simplex of Δ all the vertices lie in some element of V'. Since each open basic set is convex, the images of the vertices of Δ under g determine a piecewise linear map g' from E^n into U. We observe that $d(g, g') < \varepsilon$.

Since Δ has only finitely many vertices, we can modify the images of the vertices, each by an amount less than $[\varepsilon - d(g, g')]/10n$, to achieve a new piecewise linear mapping g'' of E^n into U such that (1) the image of each simplex of Δ is a normal finite polyhedron in some open basic set in U of diameter less than $\varepsilon/3$ and (2) $d(g, g'') < \varepsilon$. Finally, since both T and $g''(E^n)$ are normal finite polyhedrons, and since infinitely many different orthogonal directions are available, it is possible to move each vertex of $g''(\Delta)$ by an amount less than $[\varepsilon - d(g, g'')]/10n$ so that the desired piecewise linear homeomorphism g^* is achieved.

6. THE KEY LEMMA ON TRANSLATING PROPERTY Z INTO HOMEOMORPHISMS

LEMMA 6.1. *Let* $\varepsilon > 0$. *Let* K *be a closed subset of* I^∞ *such that* K *has Property* Z. *Let* M *be a standard* n*-cell in* I^∞, *and let* U *be an open set containing* M. *Then there exists a stable* β^**-homeomorphism* h *of* I^∞ *onto itself, with support on* U, *such that*

$$(1)\ M \cap h(K) = \emptyset \quad \textit{and} \quad (2)\ d(h, e) < \varepsilon .$$

Proof. Let $\tilde{M}$ be a standard cubical decomposition of M such that the mesh of $\tilde{M}$ is less than $\varepsilon/2n$. For each j $(0 \le j \le n)$, let M_j be the j-skeleton of $\tilde{M}$, and let M_j^* denote the union of the elements of M_j. We proceed by a finite inductive process. We give the first step (which is somewhat special), and the inductive step.

Let U_0 be a neighborhood of M_0 such that $U_0 \subset U$ and each component of U_0 has diameter less than $\varepsilon/2n$ and is an open basic set containing exactly one element of M_0. Since K has Property Z, K contains no open set. In each component of U_0

there is a point not in K. Therefore we may apply Lemma 5.1 to find a stable β^*-homeomorphism that (i) carries such a point onto the point of M_0 in such a component and (ii) is the identity outside such a component of U_0. A finite composition of such homeomorphisms (one for each component of U_0) is a stable β^*-homeomorphism h_0 with support on U_0 and satisfies the conditions $h_0(K) \cap M_0 = \emptyset$ and $d(h_0, e) < \varepsilon/2n$

By the inductive hypothesis, there exist stable β^*-homeomorphisms $h_0, h_1, \cdots, h_{i-1}$ with support on U such that

(1) $h_{i-1} \circ \cdots \circ h_0(K) \cap M^*_{i-1} = \emptyset$ and

(2) $d(h_j, e) < \varepsilon/2n$ for each j $(0 \leq j \leq i - 1)$.

We proceed to describe h_i. For each element $m \in M_i$, let m' be a standard i-cell with $m' \subset m$, with $m' \cap M^*_{i-1} = \emptyset$, and with

$$[\mathrm{Cl}(m \setminus m')] \cap [h_{i-1} \circ \cdots \circ h_0(K)] = \emptyset.$$

The finite collection M' of such m' (one for each element m of M_i) is a collection of disjoint standard i-cells, each of diameter less than $\varepsilon/2n$. Let U_i be an open set contained in U such that (1) U_i is the union of a finite collection V_i of disjoint open basic sets, each of diameter less than $\varepsilon/2n$, (2) for each $m' \in M'$, there exists an element v of V_i such that $m' \subset \mathrm{Int}\, v$ but $M^*_{i-1} \cap \mathrm{Cl}\, v = \emptyset$, and (3) no two elements of M_i intersect the same element of V_i.

For each $v \in V_i$, we wish to apply Lemma 5.2 followed by Lemma 5.1. Let E^i be an abstract polyhedral i-cell, and observe that $B(m')$ is a polyhedral $(i - 1)$-sphere. By Property Z, there exists a map ϕ_v of E^i into $v \setminus K$ such that $\phi_v \mid B(E^i)$ is a piecewise linear homeomorphism onto $B(m')$. Let

$$T = [\mathrm{Cl}(m \setminus m')] \cap [\mathrm{Cl}\, v],$$

and observe that $T \cap h_{i-1} \circ \cdots \circ h_0(K) = \emptyset$. By Lemma 5.2, there exists a homeomorphism ϕ^*_v of E^i onto a normal finite polyhedron in $v \setminus K$ such that

$$\phi^*_v \mid B(E^i) = \phi_v \mid B(E^i) \quad \text{and} \quad \phi^*_v(E^i) \cap (K \cup T) = B(m').$$

Let f_v be a homeomorphism of $(\mathrm{Bd}\, v) \cup T \cup \phi^*_v(E^i)$ onto $(\mathrm{Bd}\, v) \cup (m \cap v)$ that reduces to the identity on $B(v) \cup T$. By Lemma 5.1, there exists a stable β^*-homeomorphism g_v that is an extension of f_v and is supported on v. We choose $h_i = \Pi_{v \in V_i}\, g_v$ as the homeomorphism of the inductive step. Finally, $h = h_n \circ \cdots \circ h_0$ is the desired homeomorphism of the lemma.

7. FROM PROPERTY Z TO INFINITE PARTIAL DEFICIENCY

In this section we shall prove a lemma about the existence of a certain stable homeomorphism g of I^∞ onto itself. In the next section we shall use g to exhibit a stable β^*-homeomorphism translating Property Z to infinite deficiency.

LEMMA 7.1. *Let* K *be a closed subset of* I^∞ *such that* K *has Property* Z. *Then there exists a stable homeomorphism* g *of* I^∞ *onto itself such that* $g(K) \cup \mathrm{Cl}[g(s) \cap B(I^\infty)] \cup \mathrm{Cl}[g(B(I^\infty)) \cap s]$ *has infinite partial deficiency.*

The proof of Lemma 7.1 is conceptually easy but technically rather complicated. We begin with a heuristic outline of the argument employing the symbolism to be used in the proof.

(1) The homeomorphism g is to be exhibited by an inductive construction with $g = L\Pi_{\lambda>0}\, g_\lambda$ and with convergence by the convergence criterion of Lemma 2.1.

(2) Each g_λ is to be exhibited as a composition of two β^*-homeomorphisms, $g_\lambda = h_\lambda \circ f_\lambda$, where f_λ is constructed as in Lemma 6.1 to carry $g_{\lambda-1} \circ \cdots \circ g_1(K)$ off of a certain central standard n_λ-cell M_λ with M_λ chosen so that (M_λ, I^∞) is small.

(3) Since $d(f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K), M_\lambda) > 0$, there must exist a basic open set containing M_λ and not intersecting $f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)$. Thus there must exist an integer j_λ such that $j_\lambda > n_\lambda$, $j_\lambda > j_{\lambda-1}$ and for $\alpha_\lambda = \{j \mid j < j_\lambda\}$,

$$\tau_{\alpha_\lambda}(M_\lambda) \cap \tau_{\alpha_\lambda}(f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)) = \emptyset .$$

Thus $(\tau_{\alpha_\lambda}(M_\lambda) \times I_{\alpha'_\lambda}) \cap f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K) = \emptyset$.

The homeomorphism h_λ will be designed to "enlarge" a certain neighborhood of the intersection of the set $(\tau_{\alpha_\lambda}(M_\lambda) \times I_{\alpha'_\lambda})$ with the union of the two endslices in the j_λ-direction, so that $\tau_{j_\lambda} h_\lambda \circ f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K) \subset I^\circ_{j_\lambda}$. In the definitions to follow, we describe characteristics of the desired h_λ.

(4) Referring to the notation of (3), we inductively develop a sequence $(j_\lambda)_{\lambda>0}$ of positive integers. In fact, $g(K) \cup \mathrm{Cl}[g(s) \cap B(I^\infty)] \cup \mathrm{Cl}[g(B(I^\infty)) \cap s]$ is to have infinite partial deficiency with respect to $\{j_\lambda\}_{\lambda>0}$. To achieve this property it suffices, in the light of the conditions of (3), that we inductively require that for each $\lambda > 0$ there is a neighborhood V_λ of the union of the two endslices in the j_λ-direction such that $g_{\lambda'} \mid V_\lambda = e \mid V_\lambda = h_{\lambda'} \mid V_\lambda$ for each $\lambda' > \lambda$. For each $\lambda > 0$, we shall introduce a montonic increasing sequence $(J_{j_\lambda}(m))_{m>0}$ of closed subintervals of $I^\circ_{j_\lambda}$. We may consider V_λ to be $\tau^{-1}_{j_\lambda}\left(I_{j_\lambda} \setminus \bigcup_{m>0} J_{j_\lambda}(m)\right)$.

Definitions. Let i and j be integers $(i < j)$. For any point p, an (i, j)-*expansion* of $X = \{p\} \times I_i \times I_j$ is a homeomorphism h of X onto itself such that

$$h^{-1}(\{p\} \times I_i \times \{0\}) \qquad \text{and} \qquad h^{-1}(\{p\} \times I_i \times \{2^{-j}\})$$

are closed subintervals of $(\{p\} \times I^\circ_i \times \{0\})$ and $(\{p\} \times I^\circ_i \times \{2^{-j}\})$, respectively. We call the projections of these two inverse sets on I_i the *base sets* of the (i, j)-expansion. For any space Y, a (Y, i, j)-*expansion* is a homeomorphism h of $Y \times I_i \times I_j$ onto itself such that for each $y \in Y$, $h \mid (\{y\} \times I_i \times I_j)$ is either an (i, j)-expansion or the identity. For any $j > 1$, a j-*complete expansion* is a homeomorphism h of I^∞ onto itself, defined for $p \in \Pi_{k\leq j} I_k$ and $q \in \Pi_{k>j} I_k$ as $h(p, q) = (h^*(p), q)$, where h^* is a $(\Pi_{k<j, k\neq i} I_k, i, j)$-expansion for some i $(1 \leq i < j)$. We also say that h is a j-*complete expansion with respect to* i.

Proof of Lemma 7.1. We use the notation and definitions introduced above. We begin with a description of g_1. The inductive step for g_λ will differ from that for g_1 only in notation and in the necessity to handle $I_{j_1}, I_{j_2}, \cdots, I_{j_{\lambda-1}}$ rather specially.

Let $1 > \varepsilon_1 > 0$, let $n_1 \in Z$, with $2^{-n_1} < \varepsilon_1/4$, and let M_1 be a central standard n_1-cell in I^∞ such that

$$\rho(\tau_j(M_1), I_j \setminus I_j^\circ) = \varepsilon_1 \cdot 2^{-j-2} \quad \text{for each } j \ (1 \leq j \leq n_1).$$

By Lemma 6.1, there exists a β^*-homeomorphism f_1 such that $d(f_1, e) < \varepsilon_1/4$ and $f_1(K) \cap M_1 = \emptyset$. Since $f_1(K)$ and M_1 are both compact, we see that $\rho(f_1(K), M_1) > 0$. Let $\delta_1 = \rho(f_1(K), M_1)$. There exists an integer $j_1 > n_1$ such that $\sum_{j \geq j_1} 2^{-j} < \delta_1^2$. Letting α_1 denote $\{j \mid j < j_1\}$, we see that $\tau_{\alpha_1}(M_1) \cap \tau_{\alpha_1}(f_1(K)) = \emptyset$.

Now we shall define h_1 as a finite composition $h_{1,j_1-1} \circ \cdots \circ h_{1,1}$ such that $h_{1,i}$ is a j_1-complete expansion with respect to i, for each i $(1 \leq i \leq j_1 - 1)$. Specifically, for each $p \in \Pi_{k<j_1, k \neq i} I_k$, the base sets of the associated (i, j_1)-expansion of $\{p\} \times I_i \times I_{j_1}$ are both to be the closed $(\delta_1 \cdot \varepsilon_1 \cdot 2^{-j_1-2})$-neighborhood in I_i of $\tau_i(M_1)$. Also, we require that

$$d(h_{1,i}, e) < \varepsilon_1 \cdot 2^{-i-2} \text{ for } i \leq n_1 \quad \text{and} \quad d(h_{1,i}, e) < \varepsilon_1 \cdot 2^{-i+n_1-2} \quad \text{for } i > n_1.$$

Clearly, h_1 is a β^*-homeomorphism, $\tau_{j_1} h_1 \circ f_1(K) \subset I_{j_1}^\circ$, and $d(h_1 \circ f_1, e) < \varepsilon_1$. Let $(J_{j_1}(m))_{m>0}$ be a sequence of closed subintervals of $I_{j_1}^\circ$ such that

(1) $J_{j_1}(1) \supset \tau_{j_1}(h_1 \circ f_1(K))$,

(2) $J_{j_1}(m_1)$ is contained in the interior of $J_{j_1}(m_2)$ for $m_1 < m_2$, and

(3) $\bigcup_{m>0} J_{j_1}(m)$ is contained in a closed subinterval J_{j_1} of $I_{j_1}^\circ$.

We now give a description of g_λ, assuming $(g_k)_{k<\lambda}$, $(j_k)_{k<\lambda}$, and associated sets as indicated in the description of g_1. Specifically, we require inductively that $\tau_{j_k}(g_{\lambda-1} \circ \cdots \circ g_1(K)) \subset J_{j_k}(2\lambda - 3)$ for each $k < \lambda$.

Let ε_λ $(1 > \varepsilon_\lambda > 0)$ be selected so that it satisfies the convergence criterion. Let n_λ be a positive integer such that $(2^{-n_\lambda}) < \varepsilon_\lambda/4$ and $n_\lambda > j_{\lambda-1}$. Let M_λ be a central standard n_λ-cell in I^∞ such that

$$\rho(\tau_j(M_\lambda), I_j \setminus I_j^\circ) = \varepsilon_\lambda \cdot 2^{-j-2} \quad \text{for each } j \ (1 \leq j \leq n_\lambda) \text{ with } j \neq j_1, \cdots, j_{\lambda-1},$$

and $\tau_j(M_\lambda) = J_j(2\lambda - 2)$ for each j $(j = j_1, \cdots, j_{\lambda-1})$.

For $j = j_1, j_2, \cdots, j_{\lambda-1}$, let $b(j, \lambda)$ denote $\rho(I_j \setminus J_j(2\lambda - 2), J_j(2\lambda - 3))$. By definition, $b(j, \lambda) > 0$. By Lemma 6.1, there exists a β^*-homeomorphism f_λ such that

(1) $f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K) \cap M_\lambda = \emptyset$.

(2) f_λ is the identity on $\tau_j^{-1}(I_j \setminus J_j(2\lambda - 1))$ for $j = j_1, j_2, \cdots, j_{\lambda-1}$, and

(3) $d(f_\lambda, e) < \min(\varepsilon_\lambda/4, b(j_1, \lambda), \cdots, b(j_{\lambda-1}, \lambda))$.

With the definition of h_λ to follow, this will insure that $\tau_j(g_\lambda \circ \cdots \circ g_1(K)) \subset J_j(2\lambda - 1)$ for $j = j_1, \cdots, j_{\lambda-1}$. Since $f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)$ and M_λ are both compact, $\delta_\lambda = \rho(f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K), M_\lambda) > 0$. There exists an integer $j_\lambda > n_\lambda$ such that $\sum_{j \geq j_\lambda} 2^{-j} < \delta_\lambda^2$. Letting α_λ denote $\{j \mid j < j_\lambda\}$, we see that

$$\tau_{\alpha_\lambda}(M_\lambda) \cap \tau_{\alpha_\lambda}(f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)) = \emptyset.$$

Now we shall define h_λ as a finite composition $h_{\lambda,j-1} \circ \cdots \circ h_{\lambda,1}$ such that $h_{\lambda,i}$ is a j_λ-complete expansion with respect to i, for each i $(1 < i < j_\lambda - 1)$. Specifically, if $i = j_1, \cdots, j_{\lambda-1}$, then $h_{\lambda,i}$ is to be the identity. If $i \neq j_1, \cdots, j_{\lambda-1}$, then for each p in

$$\left(\prod_{\substack{k<j_\lambda \\ k\neq i, j_1, \cdots, j_{\lambda-1}}} I_k\right) \times \prod_{k=j_1, \cdots, j_{\lambda-1}} J_k(2\lambda - 1)$$

the base sets of the associated (i, j_λ)-expansion of $\{p\} \times I_i \times I_{j_\lambda}$ are both to be the closed $(\delta_\lambda \cdot \varepsilon_\lambda \cdot 2^{-2-j_\lambda})$-neighborhood in I_i of $\tau_i(M_\lambda)$; and for each $p \in \Pi_{k<j_\lambda} I_k$ with the property that $\tau_k(p) \in (I_k \setminus J_k(2\lambda))$ for some $k = j_1, \cdots, j_{\lambda-1}$, the associated (i, j_λ)-expansion is to be the identity.

We also require that $d(h_{\lambda,i}, e) < \varepsilon_\lambda \cdot 2^{-i-2}$ for $i \leq n_\lambda$ and

$$d(h_{\lambda,i}, e) < \varepsilon_\lambda \cdot 2^{-i+n_\lambda-2} \quad \text{for } i > n_\lambda.$$

Clearly, h_λ is a β^*-homeomorphism, $d(h_\lambda \circ f_\lambda, e) < \varepsilon_\lambda$, and

$$\tau_{j_\lambda}(h_\lambda \circ f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)) \subset I^\circ_{j_\lambda}.$$

Also,

$$\tau_{j_i}(h_\lambda \circ f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K)) \subset J_{j_i}(2\lambda - 1) \quad \text{for each } i \ (1 \leq i < \lambda).$$

Let $(J_{j_\lambda}(m))_{m>0}$ be a sequence of closed subintervals of $I^\circ_{j_\lambda}$ such that

(1) $\tau_{j_\lambda} h_\lambda \circ f_\lambda \circ g_{\lambda-1} \circ \cdots \circ g_1(K) \subset J_{j_\lambda}(2\lambda - 1)$,

(2) $J_{j_\lambda}(m_1)$ is contained in the interior of $J_{j_\lambda}(m_2)$ for $m_1 < m_2$, and

(3) $\bigcup_{m>0} J_{j_\lambda}(m)$ is contained in a closed subinterval J_{j_λ} of $I^\circ_{j_\lambda}$.

The homeomorphism $g = L\Pi_{\lambda>0}\, g_\lambda$ is stable, since g_1 is stable and $L\Pi_{\lambda>1}\, g_\lambda$ leaves the endslices in the j_1-direction pointwise fixed (thus making Lemma 3.2 applicable).

We note that for $\lambda > 0$ and $\lambda' > \lambda$

$$g_{\lambda'} \mid \tau_{j_\lambda}^{-1}(I_{j_\lambda} \setminus J_{j_\lambda}) = e \mid \tau_{j_\lambda}^{-1}(I_{j_\lambda} \setminus J_{j_\lambda}).$$

But since $g_\lambda \circ \cdots \circ g_1$ is a β^*-homeomorphism, we see that for each point $p \in \bigcup_{\lambda>0} \tau_{j_\lambda}^{-1}(I_{j_\lambda} \setminus J_{j_\lambda})$, $g^{-1}(p) \in s$ if and only if $p \in s$. Also, $g^{-1}(p) \notin K$. It follows that $g(K) \cup \mathrm{Cl}[g(s) \cap B(I^\infty)] \cup \mathrm{Cl}[g(B(I^\infty)) \cap s]$ has infinite partial deficiency with respect to $\{j_\lambda\}_{\lambda>0}$.

8. FROM PROPERTY Z TO INFINITE DEFICIENCY

We are now in a position to prove the following important theorem.

THEOREM 8.1. *If* K *is a closed subset of* I^∞ *and has Property* Z, *then there exists a stable* β^**-homeomorphism* f *for which the set* f(K) *has infinite deficiency in* I^∞.

Proof. By Lemma 7.1, there exists a stable homeomorphism g of I^∞ onto itself such that

$$K_0 = g(K) \cup [\mathrm{Cl}(g(s) \cap B(I^\infty))] \cup [\mathrm{Cl}(g(B(I^\infty)) \cap s)]$$

has infinite partial deficiency. By Corollary 3.4, there exists a stable β-homeomorphism h such that $h(K_0)$ has infinite deficiency. By Lemma 5.1 of [2] and Corollary 3.8 of this paper, there exists a stable β-homeomorphism θ such that $\theta^{-1}(s) = s \setminus h(K_0)$ and $\theta(h(K_0))$ has infinite deficiency. But θhg is a β-homeomorphism, and thus, by Lemma 4.3, there exists a stable homeomorphism ϕ such that $[\phi(\theta hg)]$ is a β^*-homeomorphism and $\phi\theta hg(K)$ has infinite deficiency in I^∞. It follows that $\phi\theta hg$ is the desired homeomorphism f of the lemma.

To apply Theorem 8.1 to certain subsets of I^∞ (or to s itself or to any separable infinite dimensional Fréchet space), we need a theorem of the following type. We shall give a proof based on Lemma 8.3 stated below.

THEOREM 8.2. *If* $K \subset s$ *is closed (relative to* s) *and has Property* Z *(relative to* s), *and if* $K \subset D(K) \subset I^\infty$ *and* D(K) *is closed in* I^∞ *and* $D(K) \cap s = K$, *then* D(K) *has Property* Z *(relative to* I^∞).

LEMMA 8.3. *If* U *is a nonempty, homotopically trivial open set in* I^∞, *then* $U \cap s$ *is a nonempty, homotopically trivial open set relative to* s.

Proof of Lemma 8.3. Since $I^\infty \setminus s$ contains no open set in I^∞ and the topology of s is inherited from that of I^∞, $U \cap s$ is a nonempty open set in s. Let E^n be an abstract polyhedral n-cell with boundary S^{n-1}, and let g be a mapping of S^{n-1} into $U \cap s$. By hypothesis, there exists a mapping g* of E^n into U such that $g^* \mid S^{n-1} = g$. To prove the lemma, we need to show the existence of a map h of E^n into $U \cap s$ such that $h \mid S^{n-1} = g$. Since $g^*(E^n)$ is compact, there exists a finite collection W of open basic sets such that W covers $g^*(E^n)$ and each element of W is a subset of U. Let $\triangle$ be a (closed) triangulation of E^n so fine that for each $\delta \in \triangle$, $g^*(\delta)$ is a subset of an element of W. Let $\triangle_0$ be the 0-skeleton of $\triangle$, and let h_0 be a map from $S^{n-1} \cup \triangle_0$ into U such that $h_0 \mid S^{n-1} = g$ and such that for each $p \in (\triangle_0 \setminus S^{n-1})$, $h_0(p)$ lies in the intersection of s with the intersection of the elements of W that contain $g^*(p)$. Since the intersection of s with any element of W is convex, we may realize the desired map as an extension of h_0 in a piecewise linear fashion from E^n into s.

Proof of Theorem 8.2. Let U be any nonempty homotopically trivial open set in I^∞. By Lemma 8.3, $U \cap s$ is a nonempty, homotopically trivial open set (relative to s), and $U \setminus D(K)$ is nonempty since

$$U \setminus D(K) \supset [(U \cap s) \setminus D(K)] = (U \cap s) \setminus K \neq \emptyset.$$

Let E^n be an abstract polyhedral n-cell with boundary S^{n-1}. Let g be a map of S^{n-1} into $U \setminus D(K)$, and let g* be a map of E^n into U such that $g^* \mid S^{n-1} = g$.

First we shall exhibit a map h of E^n into U such that

$$h \mid S^{n-1} = g \quad \text{and} \quad h(E^n \setminus S^{n-1}) \subset (U \cap s).$$

Since $g^*(E^n)$ is compact, there exists a finite collection W of open basic sets such that W covers $g^*(E^n)$, and each element of W is a subset of U.

Let $\triangle$ be a (closed) triangulation of E^n so fine that for each $\delta \in \triangle$, $g^*(\delta)$ is a subset of an element of W. Let $\triangle_0$ be the 0-skeleton of $\triangle$. Let h_0 be a map from $S^{n-1} \cup \triangle_0$ into U such that $h_0 \mid S^{n-1} = g$, and such that for each $p \in (\triangle_0 \setminus S^{n-1})$, $h_0(p)$ is an element of the intersection of s with the intersection of the elements of W that contain $g^*(p)$. Since each element of W is convex, we can realize the desired map h as an extension of h_0 in a piecewise linear fashion from E^n into U.

Now we complete the proof. Since $h(S^{n-1}) \cap D(K) = \emptyset$, there exists a subset E_1^n of $E^n \setminus S^{n-1}$ such that E_1^n is an n-cell and $h[\mathrm{Cl}(E^n \setminus E_1^n)] \cap D(K) = \emptyset$. Let S_1^{n-1} denote the boundary of E_1^n. Then $h(S_1^{n-1}) \subset (U \cap s)$ and $(U \cap s)$ is homotopically trivial, by Lemma 8.3. By hypothesis, $h \mid S_1^{n-1}$ can be extended to a mapping $\tilde{h}$ of E_1^n into $(U \cap s) \setminus K$. Thus the mapping that is $h \mid (E^n \setminus E_1^n)$ and is $\tilde{h}$ on E_1^n is the desired extension of g from E^n into $U \setminus D(K)$.

The following two theorems are almost immediate applications of Theorems 8.1 and 8.2. The first is also a special case of a more general theorem given in Section 10 as Corollary 10.2.

THEOREM 8.4. *If* K *is a relatively closed subset of* s *and has Property* Z *relative to* s, *then there exists a homeomorphism* f *of* s *onto itself such that* f(K) *has infinite deficiency.*

The techniques of [2] by themselves were not sufficient for proving the following theorem. However, they are applicable in the special case where the compact set M of $B(I^\infty)$ is contained in a finite union of endslices of I^∞.

THEOREM 8.5. *Let* M *be any compact subset of* $B(I^\infty)$. *Then there exists a* β-*homeomorphism* h *such that* $h(s) = s \cup M$.

Proof. Regarding K as the empty set, we see from Theorem 8.2 that M has Property Z. By Theorem 8.1, there exists a β^*-homeomorphism f such that f(M) has infinite deficiency. By Lemma 4.2, there exists a β-homeomorphism g such that $g(s) = s \cup f(M)$. The mapping $f^{-1}g$ is the desired homeomorphism h.

9. INFINITE DEFICIENCY IMPLIES PROPERTY Z

Let X be I^∞ or any separable, infinite-dimensional Fréchet space. We shall prove the following theorem.

THEOREM 9.1. *If* K *is a closed set of infinite deficiency in* X, *then* K *has Property* Z.

Proof. Let U be any nonempty, homotopically trivial open set in X. Clearly, $U \setminus K$ is open and nonempty. Let E^n be an abstract polyhedral n-cell with boundary S^{n-1}, let g be a map of S^{n-1} into $U \setminus K$, and let g^* be an extension of g from E^n into U. Since both $g^*(E^n)$ and $g^*(S^{n-1})$ are compact, there exists a finite collection W of open convex subsets of U such that W covers $g^*(E^n)$ and no element of W intersects both K and $g^*(S^{n-1})$. Let $\triangle$ be a (closed) triangulation of E^n that is so fine that

(1) $g^*(\delta) \cap K = \emptyset$ for each $\delta \in \triangle$ for which $\delta \cap S^{n-1} \neq \emptyset$, and

(2) for each $\delta \in \Delta$, $g^*(\delta)$ is contained in some element of W.

To complete the proof of Theorem 9.1, it suffices to note the existence of a map h of E^n into $U \setminus K$ such that $h \mid S^{n-1} = g$. But such an h can clearly be realized as an extension in a piecewise linear fashion of a map h_0 from the union of S^{n-1} and the 0-skeleton Δ_0 of Δ into $U \setminus K$, where $h_0 \mid S^{n-1} = g$, and where for each $p \in (\Delta_0 \setminus S^{n-1})$ the point $h_0(p)$ lies in the intersection of the elements of W that contain $g^*(p)$, and is obtained from $g^*(p)$ by small changes in finitely many coordinates with respect to which K is infinitely deficient. We may use a standard general position technique in locating the points of $h_0(\Delta_0 \setminus S^{n-1})$, so that $h(\delta) \cap K = \emptyset$ for any $\delta \in \Delta$ for which $\delta \cap S^{n-1} = \emptyset$.

10. THE HOMEOMORPHISM EXTENSION THEOREM AND ITS COROLLARIES

We are now in a position to prove our main theorem and its corollaries.

THEOREM 10.1. *Let* X *denote the Hilbert cube or any separable, infinite-dimensional Fréchet space. Let* K *be a closed subset of* X, *and let* f *be a homeomorphism of* K *onto a closed set of infinite deficiency in* X. *In order that* f *can be extended to a stable homeomorphism of* X *onto itself, it is necessary and sufficient that* K *have Property* Z.

It should be noted that the same condition applies to the compact space I^∞ and to certain linear topological spaces that are not locally compact. Also, the necessary and sufficient condition given is independent of f itself and of the topology of K. It depends only on the embedding of K in X.

Proof. Necessity. Since Property Z is (clearly) a topological property, and since f(K) has Property Z (by Theorem 9.1), it is necessary that K have Property Z.

Sufficiency. Consider first the case where $X = I^\infty$. By Theorem 8.1, there exists a stable β^*-homeomorphism h such that h(K) has infinite deficiency. Observe that $h(K) \cap B(I^\infty)$ and $f(K) \cap B(I^\infty)$ are closed relative to $B(I^\infty)$ and that each has infinite deficiency. By Lemma 4.2, there exist stable β-homeomorphisms g and ϕ such that

$$g(s) = s \cup [h(K) \cap B(I^\infty)] \quad \text{and} \quad \phi(s) = s \cup [f(K) \cap B(I^\infty)].$$

Thus $g^{-1}h$ carries $s \cup K$ onto s. Hence $g^{-1}h(K)$ is a compact subset of s. Similarly, ϕ^{-1} carries $s \cup f(K)$ onto s. Hence $\phi^{-1}f(K)$ is a compact subset of s. By Theorem 3.5, there exists a stable β^*-homeomorphism θ that extends the homeomorphism $\phi^{-1}fh^{-1}g$ of $g^{-1}h(K)$ onto $\phi^{-1}f(K)$. We see that $\phi\theta g^{-1}h$ is the desired stable homeomorphism of X onto itself that extends f.

Consider finally the case where $X \neq I^\infty$.

Since all separable infinite-dimensional Fréchet spaces are homeomorphic, there exists a homeomorphism h of X onto s. As before, we consider s to be embedded canonically in I^∞. Since Property Z is topological, since K has Property Z, and since Theorem 9.1 asserts that f(K) has Property Z, it follows that h(K) and hf(K) have Property Z with respect to s. By Theorem 8.2, Cl[h(K)] and Cl[hf(K)] have Property Z with respect to I^∞. By Theorem 8.1, there exist stable β^*-homeomorphisms ϕ and g such that g(Cl[h(K)]) and ϕ(Cl[hf(K)]) are deficient with respect to the infinite subsets α_1 and α_2 of Z, respectively. Without loss of generality, suppose $Z \setminus \alpha_1$ and $Z \setminus \alpha_2$ are also infinite. Let ρ be a β^*-homeomorphism obtained by interchanging the α_1- and $Z \setminus \alpha_2$-coordinates as well as the $Z \setminus \alpha_1$- and α_2-coordinates. Then $\rho g h(K)$ and $\phi h f(K)$ are closed sets deficient with respect to $Z \setminus \alpha_2$ and α_2, respectively.

By Theorem 4.1 of [2] (which employs Klee's method), there exists a homeomorphism η of s onto itself that extends $\phi h f h^{-1} g^{-1} \rho^{-1}$ from $\rho g h(K)$ onto $\phi h f(K)$. Therefore $h^{-1} \phi^{-1} \eta \rho g h$ is the desired extension of f from K onto f(K) (stability follows from Wong's result [9] that all homeomorphisms of s onto s are stable).

In the following two corollaries, X denotes I^∞ or any separable, infinite-dimensional Fréchet space, and K denotes an arbitrary closed subset of X.

COROLLARY 10.2. *In order that there exist a stable homeomorphism of* X *onto itself carrying* K *onto a set of infinite deficiency, it is necessary and sufficient that* K *have Property* Z.

Proof. This corollary will follow directly from Theorem 10.1, provided we can prove the existence of a closed set that has infinite deficiency and is homeomorphic to K. For the case where $X = I^\infty$, we observe that I^∞ contains many infinitely deficient subsets homeomorphic to I^∞. For $X \neq I^\infty$, it is shown independently in [6] and [8] that X contains a closed, infinite-dimensional subspace of infinite deficiency. But such a space must be homeomorphic to X, since both are separable, infinite-dimensional Fréchet spaces. Hence X does contain an infinitely deficient closed copy of K, as we wished to show.

COROLLARY 10.3. *Each homeomorphism between two closed subsets of* X *with Property* Z *can be extended to a stable homeomorphism of* X *onto itself.*

The proof uses Corollary 10.2 and Theorem 10.1, and it is obvious.

The next corollary represents a result that the author and Raymond Wong have sought for some time. It apparently does not follow directly from methods of [2]. The proof uses the full strength of the homeomorphism extension theorem, Theorem 10.1, in the form of its corollary given above.

COROLLARY 10.4. *If* h *is a homeomorphism of* I^∞ *onto itself, then* h *is stable.*

Proof. Let W be an endslice of I^∞. Then W and h(W) both have Property Z, and by Corollary 10.3 the homeomorphism $h^{-1} \mid h(W)$ carrying h(W) onto W can be extended to a stable homeomorphism g of I^∞ onto itself. But $f = gh$ is the identity on W, and thus it is stable, by Corollary 3.2. Hence $h = g^{-1} f$ is the product of two stable homeomorphisms, and thus it is stable.

We can now give an independent proof that every homeomorphism of I^∞ onto itself is isotopic to the identity, a result established originally by Wong [9] with a straightforward but rather ingenious argument.

COROLLARY 10.5. *Each homeomorphism* h *of* I^∞ *onto itself is isotopic to the identity.*

Proof. By Corollary 10.4, we may express h as $h_1 \circ \cdots \circ h_k$, where for each i $(1 \leq i \leq k)$ h_i is the identity on some open set. By use of the Alexander technique of shrinking the set of support to a point, it is clear that h_i is isotopic to the identity. Hence h is isotopic to the identity.

Finally, we observe that the (an) annulus conjecture for I^∞ has an affirmative solution. In light of the stability of all homeomorphisms of I^∞ onto itself, the proof of Lemma 9.1 of [4] (when properly interpreted for I^∞) gives the following version of the annulus theorem. We do not reproduce the proof here.

COROLLARY 10.6. *If* K_1 *and* K_2 *are disjoint closed sets in* I^∞, *and if there exist homeomorphisms* f_1 *and* f_2 *of* I^∞ *onto itself such that*

$$f_1(K_1) = f_2(K_2) = \{p \mid \tau_1(p) = 1/4\},$$

then there exists a homeomorphism f *of* I^∞ *onto itself such that*

$$f(K_1) = \{p \mid \tau_1(p) = 1/8\} \quad \textit{and} \quad f(K_2) = \{p \mid \tau_1(p) = 3/8\}.$$

11. A HOMEOMORPHISM CARRYING $B(I^\infty)$ OFF ITSELF

In this final section, we shall give the essential structure of a proof of a theorem that affords considerable insight into the role of $B(I^\infty)$ in I^∞. In spite of the fact that I^∞ has the fixed-point property, $B(I^\infty)$ can be "folded" into the pseudo-interior under a homeomorphism that is a left product of β-homeomorphisms.

THEOREM 11.1. *There exists a sequence* $(f_i)_{i>0}$ *of* β*-homeomorphisms such that* $f = L\Pi_{i>0} f_i$ *exists and* $f(B(I^\infty)) \subset s$.

For the proof we need a definition and two lemmas.

Definition. A subset K of I^∞ is said to be a j*-normal subset* of $W_j(0)$ [or of $W_j(1)$] provided $K = \Pi_{i<j} I_i \times \{p\} \times \Pi_{i>j} J_i$, where $p = 0$ [or $p = 2^{-j}$], and where for each $i > j$, J_i is a closed subinterval of I_j°.

LEMMA 11.2. *For each* $\varepsilon > 0$ *and each* $j > 0$, *there exists a* β^**-homeomorphism* g_j *such that*

(1) g_j *is supported in the* ε*-neighborhood of* $W_j(0) \cup W_j(1)$,

(2) $d(g_j, e) < \varepsilon$,

(3) $g_j(W_j(0))$ *and* $g_j(W_j(1))$ *are* j*-normal subsets of* $W_j(0)$ *and* $W_j(1)$, *respectively, and*

(4) $\beta(g_j) \supset \{1, \cdots, j-1\}$.

Proof. The proof of this lemma is basically the proof of the Contraction Theorem (Theorem 6.1 in [2]), with I_1 of [2] replaced by I_j and with the first $(j-1)$ coordinate factors held constant. We do not repeat the argument.

LEMMA 11.3. *For each* $\varepsilon > 0$ *and any* j*-normal subsets* $N_j(0)$ *and* $N_j(1)$ *of* $W_j(0)$ *and* $W_j(1)$, *respectively, there exists a* β^**-homeomorphism* h_j *such that*

(1) h_j *is supported in the* ε*-neighborhood of* $W_j(0) \cup W_j(1)$,

(2) $d(h_j, e) < \varepsilon$,

(3) $h_j(N_j(0))$ *and* $h_j(N_j(1))$ *have infinite deficiency, and*

(4) $\beta(h_j) \supset \{1, \cdots, j-1\}$.

Proof. We omit the detailed proof of this lemma, since it is basically like that of Corollary 3.4 of this paper or Lemma 3.3 of [2], where we showed how a set of infinite partial deficiency can be carried onto a set of infinite deficiency by a β^*-homeomorphism.

We are now in a position to organize a proof of Theorem 11.1. We use an inductive construction of f_i. The convergence follows from the convergence criterion of Lemma 2.1. Each f_i is to be a product $g_i^{-1} h_i^{-1} \phi_i^{-1}$ of three homeomorphisms, where g_i is as in Lemma 11.2, h_i is as in Lemma 11.3 and ϕ_i is a composition of two homeomorphisms, each as in Lemma 4.1, but independent of the first (i - 1)-coordinates. In fact, each of $\beta(g_i)$, $\beta(h_i)$, and $\beta(\phi_i)$ is to contain $\{1, \cdots, i-1\}$. We select first the homeomorphism g_i, then h_i, using the set $g_i(W_i(0) \cup W_i(1))$. Finally, we choose ϕ_i so as to move $h_i g_i [W_i(0) \cup W_i(1)]$ into

$$s \cup \Big[\bigcup_{j<i} (W_j(0) \cup W_j(1)) \Big]$$

and off of $W_i(0) \cup W_i(1)$. Then $f_i = (g_i^{-1} h_i^{-1} \phi_i^{-1})$ moves $\phi_i h_i g_i(W_i(0) \cup W_i(1))$ onto $W_i(0) \cup W_i(1)$. It is easy to verify that $(f_i \circ \cdots \circ f_1)^{-1} \left(\bigcup_{j \leq i} (W_j(0) \cup W_j(1) \right)$ must be a subset of s, and since $\beta(f_k) \supset \{1, \cdots, i\}$ for each $k \geq i$, it follows that $f(B(I^\infty) \cap B(I^\infty) = \emptyset$, as we wished to show.

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