THE FUNCTOR [, Y] AND LOOP FIBRATIONS, I.

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

There are several ways to define loop fibrations and to compare them with principal fibre bundles. Definition 1 of this paper (see Section 3) is motivated by the kind of classification theorem we obtain: If $\Omega(Y, y_0)$ is the space of loops in Y based at y_0, and X is an arbitrary topological space, then the equivalence classes (Definition 3) of our loop fibrations are in one-to-one correspondence with the homotopy classes of maps from X to Y.

The maps that we admit between loop fibrations are analogous to principal maps: A principal map restricted to a fiber of a principal bundle is essentially given by a "left translation" by an element of the group. This leads to Dold's notion of a functional bundle [1, p. 249, proof of 7.5]. The same idea can be used to define functional fibrations for loop fibrations. It is interesting that in both cases there is a "universal" function space of fiber maps that is of the same homotopy type as the "classifying" space.

For the loop fibrations, the "universal" function space is well known: it is the space of all paths in the classifying space.

2. NOTATION AND BASIC CONCEPTS

Let Y be a pathwise connected topological space. A *path* W in Y is a pair (w, r) consisting of a continuous map w: $R^+ \to Y$ (R^+ is the space of nonnegative real numbers) and a number r in R^+ such that $w(t) = w(r)$ whenever $t \geq r$. The space of paths in Y is defined by

$$MY = \{W \mid W \text{ is a path in } Y\}.$$

Its topology is the subspace topology of $Y^{R^+} \times R^+$, Y^{R^+} having the compact-open topology.

Let $W_1 = (w_1, r_1)$ and $W_2 = (w_2, r_2)$ be paths in Y such that $w_1(r_1) = w_2(0)$. We define the sum $\mu(W_1, W_2) = W_1 + W_2 = (w_1 + w_2, r_1 + r_2)$ of the two paths by the formula

$$(w_1 + w_2)(t) = \begin{cases} w_1(t) & (0 \leq t \leq r_1), \\ w_2(t - r_1) & (r_1 \leq t). \end{cases}$$

The addition is not commutative, but it is continuous and associative whenever defined.

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For $y_0 \in Y$, let

$$E(Y, y_0) = \{W \mid W = (w, r) \in MY,\ w(0) = y_0\}$$

and

$$\Omega(Y, y_0) = \{W \mid W = (w, r) \in MY,\ w(0) = w(r) = y_0\}.$$

The spaces E and Ω both obtain the subspace topology from MY. The partial multiplication of MY makes $\Omega(Y, y_0)$ into an associative H-space with unit element $W_0 = (y_0, 0)$. (We denote the constant map from R^+ to y_0 by y_0.) The partial multiplication of MY also induces an action $\Omega(Y, y_0) \times E(Y, y_0) \to E(Y, y_0)$.

The map ν: $MY \to MY$, defined by the formula $\nu(W) = -W = (-w, r)$, where

$$-w(t) = \begin{cases} w(r - t) & (0 \leq t \leq r), \\ w(0) & (r \leq t), \end{cases}$$

gives rise to a homotopy inverse in $\Omega(Y, y_0)$ in the obvious fashion. We need the following basic properties.

LEMMA 1. *The maps* p_M: $MY \to Y$ *and* p_E: $E(Y, y_0) \to Y$ *given by the formulas*

$$p_M(w, r) = w(0) \quad \textit{and} \quad p_E(w, r) = w(r)$$

are Hurewicz fibrations.

As an example we give the proof of the first property: Let $X \times I \xrightarrow{\bar{h}} Y$ be a map, and let H_0: $X \to MY$ be such that $p_M H_0(x) = \bar{h}(x, 0)$. Then $\bar{h}$ gives rise to a map from X to MY: For each $\tau \in I$, consider $\overline{H}_\tau(x) = (\bar{h}_+(x, \tau - t), \tau)$, where

$$\bar{h}_+(x, \tau - t) = \begin{cases} \bar{h}(x, \tau - t) & (0 \leq t \leq \tau), \\ \bar{h}(x, 0) & (\tau \leq t). \end{cases}$$

Then $H(x, \tau) = \overline{H}_\tau(x) + H_0(x)$ is a covering homotopy.

We observe that this covering homotopy is equivalent to a path in the space of crossections of a fiber space over X induced by $\bar{h}_0$ from MY.

LEMMA 2. *If* $U \subset Y$ *is contractible, then* $p_M^{-1} U$ *is fiber-homotopy equivalent to* $U \times E(Y, y_0)$.

Proof. Since Y is pathwise connected, we may assume that U is contractible to y_0; let $\bar{k}$: $U \times I \to Y$ be a contraction of U to y_0, and let

$$\bar{k}_+(y, t) = \begin{cases} \bar{k}(y, t) & (0 \leq t \leq 1), \\ \bar{k}(y, 1) = y & (1 \leq t). \end{cases}$$

Now define the map K: $U \to MY$ by the formula $K(y) = (\bar{k}_+(y, t), 1)$. Then the maps

$$\alpha(W) = (p_M W, K(p_M W) + W) \qquad (W \in p_M^{-1} U),$$

$$\beta(y, W) = -K(y) + W \qquad (W \in E(Y, y_0))$$

form a fiber-homotopy equivalence $\alpha: p_M^{-1} U \rightleftarrows U \times E(Y, y_0): \beta$.

Remark. Similarly,

$$\alpha(W) = (W - K(p_E W), p_E W) \qquad (W \in E(Y, y_0)),$$

$$\beta(W, y) = W + K(y) \qquad (W \in \Omega(Y, y_0)),$$

for a fiber-homotopy equivalence between $\Omega(Y, y_0) \times U$ and $p_E^{-1} U$. Both fiber-homotopy equivalences are equivalent to crossections in the fibrations $p_M \mid p_M^{-1} U$ and $p_E \mid p_E^{-1} U$, respectively, and the suitable homotopies form paths in the corresponding space of crossections. Therefore in the case of p_E the maps α and β are compatible with the action of $\Omega(Y, y_0)$; for example, the diagram

$$\begin{array}{ccc} \Omega(Y, y_0) \times \Omega(Y, y_0) \times U & \xrightarrow{1\times\beta} & \Omega(Y, y_0) \times p_E^{-1} U \\ \downarrow \mu\times 1_U & & \downarrow \mu_E \\ \Omega(Y, y_0) \times U & \xrightarrow{\beta} & p_E^{-1} U \end{array}$$

is commutative.

3. THE THEOREMS

Definition 1. For a given based space (Y, y_0), a *loop-fibration* is a quintuple (E, p, X, f, Y), where X is a topological space, $f: X \to Y$ is a map, and (E, p, X) is the fiber space induced by f from $(E(Y, y_0), p_E, Y)$. In general, we shall use the notation (E_f, p, X).

In order to get a category of loop fibrations with respect to (Y, y_0), let us look for fiber maps that are compatible with the action of $\Omega(Y, y_0)$ on the loop fibrations. Restricted to a fiber, such a fiber map is simply a map from one fiber of $E(Y, y_0)$, say $p_E^{-1}(y_1)$, to another fiber of $E(Y, y_0)$, say $p_E^{-1}(y_2)$, since we consider only induced fibrations. A path from y_1 to y_2, that is, any element of $p_M^{-1}(y_1)$, provides such a map: the compatibility with the loop action is automatic.

We shall use (MY, p_M, Y) as a "universal" functional fibration. For any two loop fibrations with respect to (Y, y_0), say (E_f, p_1, X_1) and (E_g, p_2, X_2), we define a functional fibration $(M_{f,g}, p, X_1)$ as follows. Let

$$M_g Y = \{W \mid w(r) \in g(X_2)\} \qquad (W = (w, r) \in MY);$$

then

$$M_{f,g} = \{(W, x) \mid W \in M_g Y,\ x \in X_1, \text{and } p_M W = f(x)\};$$

that is, $(M_{f,g}, p, X_1)$ is induced by f from $(M_g Y, p_M \mid M_g Y, Y)$.

Definition 2. A fiber map $k: E_f \to E_g$ (inducing $\bar{k}: X_1 \to X_2$) is a *loop-fiber map* if there exists a crossection $\sigma: X_1 \to M_{f,g}$ $(\sigma(x) = (S(x), x))$ such that $k(W, x) = (W + S(x), \bar{k}(x))$ $((W, x) \in E_f)$.

Definition 3. Two loop fibrations with the same base space are *equivalent*, if they are fiber-homotopy equivalent through loop-fiber maps.

We notice that if $U \subset Y$ is contractible to y_0, then $p_E^{-1} U$ and $\Omega(Y, y_0) \times U$ are equivalent as loop fibrations. We may consider

$$(p_E^{-1} U, p_E \mid p_E^{-1} U, U) \quad \text{as induced by } i_1\colon U \subset Y$$

and

$$(\Omega(Y, y_0) \times U, pr_2, U) \quad \text{as induced by } i_2\colon U \to y_0 \in Y.$$

The equivalence of these two loop fibrations is due to the fact that i_1 and i_2 are homotopic:

THEOREM 1. *If* $f, g\colon X \to Y$ *are homotopic maps, then* (E_f, p, X) *and* (E_g, p, X) *are equivalent as loop fibrations.*

Proof. Let $k\colon X \times R^+ \to Y$ be such that $k(x, 0) = f(x)$ and $k(x, t) = g(x)$ for $t \geq 1$. Then $K(x) = (k(x, t), 1)$ defines a map from X into $M_g Y$. Let

$$\alpha(W, x) = (W + K(x), x) \quad \text{for } (W, x) \in E_f,$$

$$\beta(W, x) = (W - K(x), x) \quad \text{for } (W, x) \in E_g;$$

clearly, α and β form an equivalence of loop fibrations.

We now assume that Y admits a numerable covering $\mathcal{U}$ of open, contractible sets.

THEOREM 2. *If* (E_f, p, X) *and* (E_g, p, X) *are equivalent loop fibrations with respect to* (Y, y_0), *then* f *and* g *are homotopic.*

Theorems 1 and 2 together form our classification theorem.

Proof (due to D. Puppe). An equivalence between (E_f, p, X) and (E_g, p, X) is given by a crossection $\sigma\colon X \to M_{f,g}$ and a fiber map $k\colon E_f \to E_g$.

If S denotes the composition $X \xrightarrow{\sigma} M_{f,g} \subset MY \times X \xrightarrow{\text{proj}} MY$, then we require that $k(W, x) = (W + S(x), x)$. Associated with $S\colon X \to MY$ we have the mapping $\hat{S}\colon X \times R^+ \to Y$, which can be extended to $X \times \hat{R}^+$ ($\hat{R}^+$ is the one-point compactification of R^+; since for each $x \in X$, $S(x)\colon R^+ \to Y$ is constant for large $t \in R^+$, the extension is continuous, because $\{(x, t) \mid t \geq r(x)\}$ is closed). $\hat{S}(x, 0) = f(x)$, $\hat{S}(x, \infty) = g(x)$, and so f and g are homotopic.

Theorem 2 can also be proved by the methods of A. Dold (see [1]). I think this proof is of independent interest. We have to assume that Y admits a numerable covering $\mathcal{U}$ of open sets that are contractible in Y. We first establish the following *property of loop fibrations:* Let (E_f, p, X) be a loop fibration with respect to (Y, y_0), and let $A \subset X$ be a set that admits a halo H (see [1]) in X. If $k_A\colon p^{-1} A \to E(Y, y_0)$ is a loop-fiber map that can be extended as a loop-fiber map to $p^{-1} H$, then it can be extended to E_f (as a loop-fiber map, of course).

Proof. $(E(Y, y_0), p_E, Y)$ is induced from $(E(Y, y_0), p_E, Y)$ by 1_Y. The functional fibration $(M_{f,1_Y}, p_f, X)$ ((M_f, p_f, X) for short) is therefore induced from (MY, p_M, Y) by the map $f\colon X \to Y$.

The loop-fiber map k_A corresponds to a crossection $\sigma_A\colon A \to p_f^{-1} A$ that can be extended to $\sigma_H\colon H \to p_f^{-1} H$ in such a way that $k_H(W, x) = W + S_H(x)$ extends k_A. Note that $\sigma_H(x) = (S_H(x), x)$, where $S_H(x) \in MY$.

Let $\mathscr{U}$ be a numerable covering of Y by open, contractible sets; then, by Lemma 2, $p_M^{-1} U$ is fiber-homotopy equivalent to $U \times E(Y, y_0)$ for each $U \in \mathscr{U}$. The family $\mathscr{V} = \{f^{-1} U \mid U \in \mathscr{U}\}$ is an open, numerable covering of X such that $p_f^{-1}(f^{-1} U)$ is fiber-homotopy equivalent to $f^{-1} U \times E(Y, y_0)$. Since $E(Y, y_0)$ is contractible, σ_A can be extended to $\sigma: X \to M_f$, according to [1, Corollary 2.8, p. 229]. If $\sigma(x) = (S(x), x)$, then $k(W, x) = W + S(x)$ is an extension of k_A.

Second proof of Theorem 2. Assume (E_f, p, X) and (E_g, p, X) are equivalent as loop fibrations. Let $\alpha: E_f \rightleftarrows E_g: \beta$ be equivalences. Consider $(E_f \times I, p \times 1_I, X \times I)$. This is a loop fibration, induced by $h: X \times I \to Y$, $h(x, t) = f(x)$ for all $t \in I$. Let $A = X \times \dot{I}$ and $k_A: E_f \times \dot{I} \to E(Y, y_0)$ be defined as

$$k_A(W, x, 0) = W \quad \text{and} \quad k_A(W, x, 1) = G \cdot \alpha(W, x),$$

where $G: E_g \to E(Y, y_0)$ is the loop fiber map induced by g. Since $X \times \dot{I}$ admits a halo in $X \times I$, say $H = X \times [0, 1/2) \cup X \times (1/2, 1]$, and since k_A can be extended to $(p \times 1_I)^{-1} H$, k_A can be extended to $X \times I$, thus inducing a homotopy between f and g.

REFERENCE

1. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) 78 (1963), 223-255.

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