SMOOTHNESS CONDITIONS ON CONTINUA IN EUCLIDEAN SPACE

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Dedicated to R. L. Wilder on his seventieth birthday.

One would like an easy way to identify manifolds in the set of topological spaces, and as a step in this direction one may restrict attention to continua in euclidean spaces and use properties of their imbedding as well as intrinsic topological properties. This note is a report on efforts to identify (n - 1)-manifolds in R^n on the basis of both intrinsic and positional properties.

1. FREE CONTINUA

In 1933, Borsuk [5] defined a subset M of R^n to be *free* if for each $\varepsilon > 0$ there exists a continuous map $f: M \to R^n$ such that each point is moved a distance less than ε and $f(M) \cap M = \emptyset$. He asked whether a free locally contractible continuum M that separates R^n must be an (n - 1)-manifold. It is clear that local contractibility (or some form of local connectedness) is needed, because the Warsaw Circle (Figure 1) is free in R^2 but is surely not a manifold.

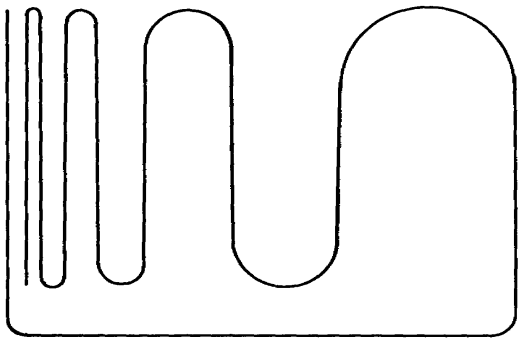

Figure 1

In the same year, Wilder [19] showed that such an M in R^2 or R^3 is a manifold (of dimension one or two, respectively). For the higher dimensions he proved that M is an (n - 1)-dimensional generalized closed (homology) manifold. This means that homologically it has the local properties of a manifold. Here is Wilder's original definition (in which the geometrical meaning of the definition is more apparent than in more recent definitions based on cohomology and sheaf theory).

We use Čech homology with coefficients in a field F. The space X (assumed to be separable and metric, for convenience) is an n-dimensional generalized manifold if it has covering dimension n and satisfies the following four conditions.

(i) $H_n(X; F) \cong F$, but if Y is any proper closed subset of X, then $H_n(Y; F) = 0$.

(ii) All sufficiently small cycles bound.

(iii) For each $x \in X$ and each $\varepsilon > 0$, there exist δ and η such that $0 < \eta < \delta < \varepsilon$ and each i-cycle $(1 \leq i \leq n - 2)$ on the "sphere"

$$S(x, \delta) = \{y \mid \rho(x, y) = \delta\}$$

bounds in the "annulus" $B(x, \varepsilon) - B(x, \eta)$.

(iv) An (n - 1)-cycle on $S(x, \delta)$ bounds in $X - B(x, \eta)$.

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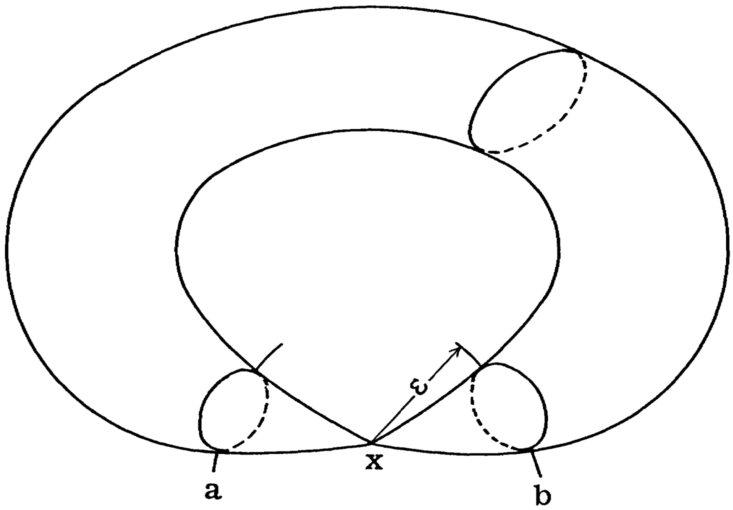

Figure 2

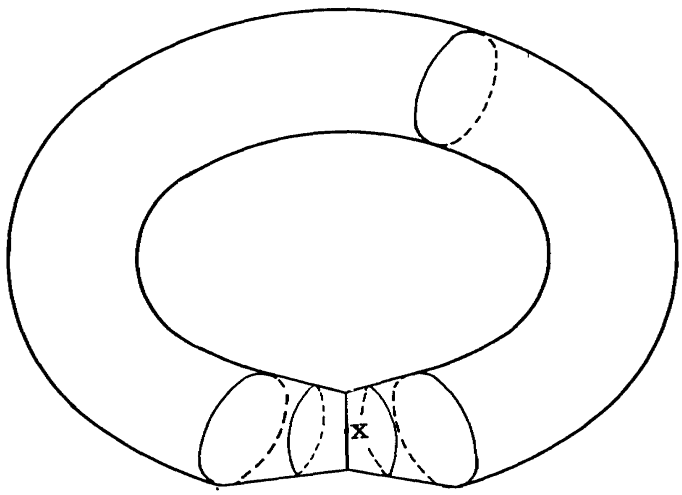

Figure 3

A pinched torus (Figure 2) is not a generalized manifold, since it violates (iii). In particular, the points a and b in the figure constitute a 0-cycle that does not bound in B(x, ε) - x. In general, this condition eliminates "local cut points" that cut cycles of any dimension. Condition (iv) implies that a torus pinched by the reduction of a meridian into an arc (Figure 3) is not a generalized manifold.

It is easy to verify that triangulated generalized n-manifolds have the property that the link of each vertex is an homology (n - 1)-sphere. It is known that separable metric generalized manifolds of dimensions one and two are topological manifolds [22], and that generalized manifolds that are simplicial complexes of dimension three are topological manifolds [15], [9]. The 4-dimensional simplicial complex obtained by suspending the dodecahedral space [17, p. 215] is a generalized manifold, but is not a topological manifold.

Wilder's result cited above on locally contractible free continua is the "best possible". There exists an example, due to Newman [16], of a free 4-dimensional generalized manifold M that separates S^5 but is not a manifold (it is not even locally simply connected). Similar examples in S^4 are given in [8], and higher-dimensional examples follow easily from these.

2. UNIFORM LOCAL CONNECTEDNESS

Instead of imposing the condition of freeness on the continuum M in R^n, one may impose conditions on the domains complementary to M, in the hope of guaranteeing that M be a manifold. Brouwer [6], [7] showed that each (n - 1)-manifold M in R^n separates R^n into two domains, and that each point of M is arcwise accessible from each domain. It turns out that the converse is true in R^2; namely, if M is a continuum separating R^2 into two domains and each point of M is arcwise accessible from each domain, then M is a 1-sphere. However, the Ann Arbor Sphere S (Figure 4) in R^3 shows that the result does not generalize, because S is not a manifold.

Wilder [18] noted that the bounded domain of R^3 - S is not uniformly locally connected (0-ulc), and he proved that if M is a common boundary of two domains in

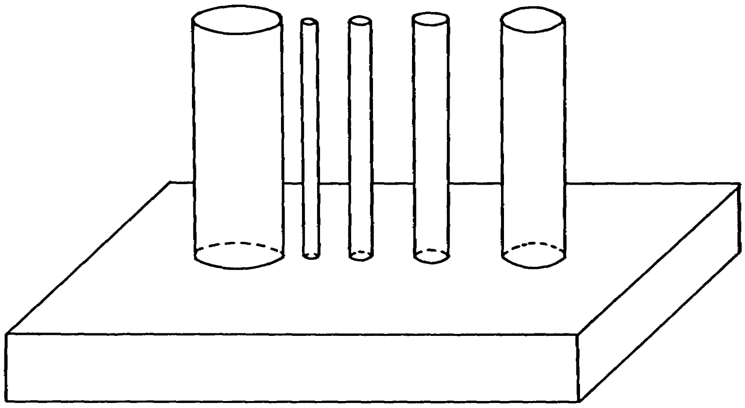

Figure 4

R^3 and these domains are 0-ulc, then M is a 2-manifold. He showed [20], [21] that the generalization to higher dimensions again leads to generalized (homology) manifolds: If the complementary domains D_1 and D_2 are i-ulc for $0 \leq i \leq n_k$ (k = 1, 2), where $n_1 + n_2 = n - 3$, then M is a generalized manifold. (In addition, we must require that small $(n_1 + 1)$-cycles in D_1 bound in D_1.)

3. HOMOTOPY MANIFOLDS

A continuum M separating S^n is *free by deformation* into a complementary domain A if there exists a homotopy $h: M \times I \to S^n$ such that h_0 is the identity and $h_t(M) \subset A$ for $t > 0$. Wilder [19] showed that in this situation M is a closed (n - 1)-generalized manifold. Eilenberg and Wilder [11] showed that M is free by deformation into A if $M = \partial A$ and A is uniformly locally contractible (ULC). Newman's example cited in Section 1 shows that even A being ULC is not sufficient to make M a manifold, for M may fail to be locally simply connected. Therefore it seems natural to require that M be locally contractible and A be ULC. This is still insufficient to make M a manifold. Curtis and Wilder [10] showed that collapsing a wild arc α in S^3 can give a locally contractible generalized manifold that is not a manifold. It follows from results of Andrews and Curtis [1] that $S^3/\alpha \times R$ is a manifold, and the complementary domains of $S^3/\alpha \times 0$ are surely ULC.

The trouble with S^3/α is that it is not like a manifold homotopically. H. B. Griffiths [12] defined homotopy manifolds by imposing a condition saying roughly that the annular region about each point must have the homotopy type of a sphere (of the appropriate dimension). Homotopy manifolds are also homology manifolds. But Curtis and Wilder [10] showed that the Bing dog-bone space B [2] is such a homotopy manifold, whereas Bing [2] had shown it is not a manifold. Also, Bing [3] showed that $B \times R = R^4$, so that M being a homotopy manifold and the complementary domains being ULC is still insufficient to ensure that M is a manifold.

This about ends the story on homotopy manifolds. Eilenberg and Wilder [11] asked whether M being free by deformation into a complementary domain A implies that A is ULC, and this question remains unanswered in general. Hempel [13] has proved that for 3-space the answer is affirmative. Other kinds of homotopy manifolds have been defined [14], [9], but they are about the same as those of Griffiths. No example is known of a polyhedral homotopy manifold that is not a manifold. Three-dimensional polyhedral homotopy manifolds are manifolds (since they are homology generalized manifolds), and four-dimensional ones are manifolds if the 3-dimensional Poincaré conjecture is true.

4. PARTIALLY FREE CONTINUA

In 1959, Bing [4] proved that any 2-sphere S^2 in 3-space is *almost* free in the sense that it can be ε-transformed by a homeomorphism into a complementary domain, except that it may leave a Cantor set T in S^2 fixed. Wilder [23] showed that each locally connected continuum M in 3-space having this almost-free property is a 2-manifold. He generalized this to n-space, showing that M is an (n - 1)-generalized manifold. In the generalization, T may be much more general than a Cantor set; it is simply required to satisfy some homology conditions. Also, the ε-transformation need not be a homeomorphism. The most general form of these results was given by Wilder in 1961 [24].

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