WILD CELLS AND SPHERES IN HIGHER DIMENSIONS

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

The purpose of this paper is to apply a theorem of Andrews and Curtis [1] to get a rapid formula for constructing wild k-cells and k-spheres in S^n. In Section 4 we construct an arc in S^n $(n > 3)$ that pierces no locally flat (n - 1)-sphere. (The somewhat lengthy interval between discovery and publication has led to the prior appearance of applications of and reference to this technique in the literature [9], [11].) Our starting point is the following obvious modification of the results of [1]:

THEOREM (Andrews and Curtis). *Let α be an arc in S^n. Then the suspension $\sigma(S^n/\alpha)$ of the quotient space S^n/α is homeomorphic to S^{n+1}.* (If X is compact, we use $\sigma(X)$ to denote the quotient space of $X \times [0, 1]$ obtained by pinching $X \times 0$ and $X \times 1$ to points.)

2. THE CONSTRUCTION α^*

Let α be an arc in S^n, and π the projection map $\pi: S^n \to S^n/\alpha$. This induces the natural suspensions $\sigma(\pi): \sigma(S^n) \to \sigma(S^n/\alpha)$, where the image and domain spaces are both S^{n+1}. Let $\alpha^* = \sigma(\pi(\alpha)) \subset \sigma(S^n/\alpha)$ be the suspension of the point $\langle \alpha \rangle$ of S^n/α. Then α^* is an arc and $\sigma(\pi) \mid \sigma(S^n) - \sigma(\alpha)$ is a homeomorphism onto $\sigma(S^n/\alpha) - \alpha^*$. On the other hand, $\sigma(S^n) - \sigma(\alpha)$ is homeomorphic to $(S^n - \alpha) \times R'$, since $\sigma(\alpha)$ contains the suspension points. Hence

(2.1) $\sigma(S^n/\alpha) - \alpha^*$ *is homeomorphic to* $(S^n - \alpha) \times R'$,

(2.2) *for every arc $\alpha \subset S^n$ there is an arc $\alpha^* \subset S^{n+1}$ such that $S^n - \alpha$ and $S^{n+1} - \alpha^*$ have the same homotopy type,*

(2.3) *for each $n \geq 3$ there exists an arc in S^n whose complement is not simply connected.*

We get (2.3) by repeated applications of (2.2) to the arc (1.1) of [8].

(2.4) *For each pair (n, k) with $n \geq 3$ and $1 \leq k \leq n$, there exists a k-cell in S^n whose complement is not simply connected.*

Proof. Let P(n, k) denote the statement of (2.4) for a fixed admissible pair (n, k), and P(n, *) the statement for n fixed and all admissible k. P(3, *) is proved in [8]. Inductively, suppose P(n, *) is true. From (2.3) we have (n + 1, 1). But if $k > 1$, then P(n + 1, k) follows from P(n, k - 1). For if α^{k-1} is a (k - 1)-cell in S^n and $\pi_1(S^n - \alpha^{k-1})$ is nontrivial, then $\alpha^k = \sigma(\alpha^{k-1})$ is a k-cell in $S^{n+1} = \sigma(S^n)$. Since $\sigma(\alpha^{k-1})$ contains the suspension points, $S^{n+1} - \alpha^k$ is homeomorphic to $(S^n - \alpha^{k-1}) \times R'$.

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3. SOME PRELIMINARY LEMMAS

Let $B \subset A \subset X$, where X is a topological space. Then X - A is k-*connected* at B if for each neighborhood U of B there exists a neighborhood V of B such that $V \subset U$ and $\pi_i(V - A)$ is trivial for $0 \leq i \leq k$. The set X - A is *projectively* k-*connected* at B if each neighborhood U of B contains a neighborhood V of B such that the induced maps i: $\pi_i(V - A) \to \pi_i(U - A)$ are trivial for $0 \leq i \leq k$.

(3.1) LEMMA. $(X/A) \times R' - \langle A \rangle \times R'$ *is projectively* k-*connected at* $\langle A \rangle \times \frac{1}{2}$ *if and only if* X - A *is projectively* k-*connected at* A. ($\langle A \rangle$ denotes the point determined by A in the quotient space X/A.)

Proof. Suppose X - A is projectively k-connected at A. Let U be a neighborhood of $\langle A \rangle \times \frac{1}{2}$ in $(X/A) \times R'$. Then there is a neighborhood W of A and an $\varepsilon > 0$ such that

$$(W/A) \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \subset U.$$

Let V be a neighborhood of A such that $V \subset W$ and $i_*: \pi_i(V - A) \to \pi_i(W - A)$ is trivial for $0 \leq i \leq k$. Then $(V/A) \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$ is the required neighborhood of $\langle A \rangle \times \frac{1}{2}$. Conversely, suppose $(X/A) \times R' - \langle A \rangle \times R'$ is projectively k-connected at $\langle A \rangle \times \frac{1}{2}$. Let U be a neighborhood of A. Then $(U/A) \times R'$ is a neighborhood of $\langle A \rangle \times \frac{1}{2}$. Hence there exists a neighborhood W of $\langle A \rangle \times \frac{1}{2}$ such that $W \subset (U/A) \times R'$ and

$$i_*: \pi_i(W - \langle A \rangle \times R') \to \pi_i((U/A) \times R' - \langle A \rangle \times R') \quad \text{is trivial } (0 \leq i \leq k).$$

Let V be a neighborhood of A, and choose ε small enough so that

$$\langle A \rangle \times \frac{1}{2} \subset (V/A) \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \subset W.$$

Then V is the required neighborhood of A.

(3.2) LEMMA. *Let* X *be compact and* $A \subset X$. *If* $\sigma(X/A) - \sigma(\langle A \rangle)$ *is projectively* k-*connected at* $\sigma(\langle A \rangle)$, *then* X - A *is* k-*connected.* (For the definition of σ, see the end of Section 1.)

Proof. Clearly it suffices to prove that $(X/A) \times \frac{1}{2} - \langle A \rangle \times \frac{1}{2}$ is k-connected. But $\left((X/A) \times \frac{1}{2} - \langle A \rangle \times \frac{1}{2}\right) \times R'$ is homeomorphic to $\sigma(X/A) - \sigma(\langle A \rangle)$, so it will suffice to prove that $\sigma(X/A) - \sigma(\langle A \rangle)$ is k-connected. Let

$$f: S^i \to \sigma(X/A) - \sigma(\langle A \rangle).$$

Since S^i is compact, $f(S^i) \subset (X/A) \times [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. By hypothesis, there exists a neighborhood V of $\sigma(\langle A \rangle)$ such that

$$i_*: \pi_i(V - \sigma(\langle A \rangle)) \to \pi_i(\sigma(X/A) - \sigma(\langle A \rangle))$$

is trivial. Since X/A is compact, there is a $t_0 < 1$ such that $(X/A) \times t \subset V$ whenever $t_0 \leq t \leq 1$. Let H be a homotopy of $\sigma(X/A) - \sigma(\langle A \rangle)$ into itself that slides $(X/A) \times \varepsilon$ "vertically" up into $(X/A) \times t_0$. This homotopy carries $f(S^i)$ into $V - \sigma(\langle A \rangle)$. Hence $f(S^i)$ bounds in $\sigma(X/A) - \sigma(\langle A \rangle)$.

(3.3) LEMMA. *Let* X *be compact, let* $A \subset X$, *and let* a *be a point of* A. *If* $\sigma(X) - \sigma(A)$ *is projectively* k-*connected at* $a \times \frac{1}{2}$, *then* X - A *is projectively* k-*connected at* a.

Proof. Let U be a neighborhood of a in X. By hypothesis, there exists a neighborhood W of $a \times \frac{1}{2}$ such that $W \subset U \times \left(\frac{1}{4}, \frac{3}{4}\right)$ and

$$i_*: \pi_i(W - \sigma(A)) \to \pi_i\left(U \times \left(\frac{1}{4}, \frac{3}{4}\right) - \sigma(A)\right)$$

is trivial $(0 \leq i \leq k)$. Without loss of generality, we may assume that $W = V \times \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$ for some neighborhood V of a and some $\delta > 0$. It is easy to see that $i_*: \pi_i(V - A) \to \pi_i(U - A)$ is trivial $(0 \leq i \leq k)$.

4. SOME SPECIAL CONSTRUCTIONS

(4.1) *For* $n > 3$, *there exists an arc* α_n *in* S^n *such that*

(1) α_n *is wild at each point,*

(2) α_n *is not cellular,*

(3) *every proper subarc of* α_n *is cellular (and wild).*

(Recall that an arc is *wild at a point* if it is not locally flat at the point, and that a subset A of an n-manifold is *cellular* if for each neighborhood U of A there exists an n-cell Q^n such that $A \subset \mathring{Q}^n \subset Q^n \subset U$.)

Proof. Let α_3 be the arc (1.1) of [8]. Then $S^3 - \alpha_3$ is not 1-connected, and $S^3 - \alpha_3$ is not projectively 1-connected at α_3. Therefore $\alpha_4 = \alpha_3^*$ (see Section 2) is an arc in S^4 such that $S^4 - \alpha_4$ is not 1-connected (by 2.1) and not projectively 1-connected at α_4 (by 3.2). Similarly, $\alpha_5 = \alpha_4^*$ inherits these two properties, and so forth. For $n > 3$, $\alpha_n = \alpha_{n-1}^*$. Since $S^{n-1} - \alpha_{n-1}$ is not projectively 1-connected at α_{n-1}, it follows from (3.1) that $S^n - \alpha_n$ is not projectively 1-connected at any interior point of α_n. Hence α_n is everywhere wild. Since $S^n - \alpha_n$ is not 1-connected, α_n is not cellular. The fact that every proper subarc of α_n $(n > 3)$ is cellular is a consequence of the following observation.

(4.2) *If* α *is any arc in* S^n, *then every proper subarc of* α^* *is cellular.*

This is a special case of a collection of theorems about spaces whose cones are euclidean at the cone point. For a proof of a theorem implying (4.2), see Rosen [10].

(4.3) *For* $n \geq 4$, *there exist a* 1-*sphere* Σ_n^1 *and a point* $P_n \in \Sigma_n^1$ *such that* $S^n - \Sigma_n^1$ *is not projectively* 1-*connected at* P_n (hence Σ_n^1 is wild).

Proof. For $n > 3$, let α_{n-1} be an arc in S^{n-1} such that $S^{n-1} - \alpha_{n-1}$ is not projectively 1-connected at α_{n-1}. Let q_{n-1} be a point of S^{n-1}/α_{n-1} other than $\langle \alpha_{n-1} \rangle$. Then $\Sigma_n^1 = \sigma(\langle \alpha_{n-1} \rangle \cup q_{n-1})$ is a 1-sphere in $S^n = \sigma(S^{n-1}/\alpha_{n-1})$, and by (3.1), $S_n - \Sigma_n^1$ is not projectively 1-connected at $P_n^k = \left\langle \alpha_{n-1} \times \frac{1}{2} \right\rangle$.

(4.4) *Let* $n \geq 3$, $1 \leq k < n$, $n - k \neq 2$. *Then there exist a* k*-sphere* Σ_n^k *in* S^n *and a point* $P_n \in \Sigma_n^k$ *such that* $S^n - \Sigma_n^k$ *is not projectively* 1*-connected at* P_n (and since $n - k \neq 2$, Σ_n^k is wild).

Proof. For $n = 3$, we have $k = 2$, and the appropriate example is described in [8]. Suppose we have constructed a Σ_n^k, P_n^k for all admissible (n, k) with $n < n_0$. Then by (4.3) we may assume $k > 1$. But then let $\Sigma_{n_0}^k = \sigma\left(\Sigma_{n_0-1}^{k-1}\right)$, and let $P_{n_0}^k = P_{n_0-1}^{k-1} \times \frac{1}{2}$ in the space $\sigma(S^n) = S^{n+1}$. Since $S^{n_0-1} - \Sigma_{n_0-1}^{k-1}$ is not projectively 1-connected at $P_{n_0-1}^{k-1}$, it follows from (3.3) that $S^{n_0} - \Sigma_{n_0}^k$ is not projectively 1-connected at $P_{n_0}^k$. This completes the proof.

The only case not covered by (4.3) and (4.4) is that in which $n - k = 2$. But here we may choose, in S^3, the simple closed curve (2.1) of [8] whose complement has a nonabelian fundamental group. Various suspensions of the 3-sphere and the wild simple closed curve produce all the required examples for co-dimension 2.

(4.5) *An arc in* S^n *(for* $n > 3$*) that pierces no locally flat* $(n - 1)$*-sphere.*

An arc α in S^n pierces a sphere Σ^{n-1} at a point P if for some subarc β of α, $\beta \cap \Sigma^{n-1} = P$ and the endpoints of β are in different components of $S^n - \Sigma^{n-1}$. The arcs of (4.1) satisfy (4.5). In fact, we shall prove the following.

(4.6) THEOREM. *If* $\alpha \subset S^n$ $(n > 3)$ *is a noncellular arc such that every proper subarc of* α *is cellular, then* α *pierces no locally flat* $(n - 1)$*-sphere.*

For $n = 2$ the theorem is vacuous, and for $n = 3$ it is false. In order to prove (4.6), we shall need the following lemmas. Since the proofs of the first two lemmas are similar to those in [4], we shall only outline them. The third lemma is an application of a theorem of Cantrell.

LEMMA 1. *Suppose* B^n *is an* n*-cell,* $A \subset B^n$, *and* $A \cap \dot{B}^n$ *is a single point* a. *Suppose also that* B^n/A *is homeomorphic to* B^n. *Then there exists a map* $f: B^n \to B^n$ *such that* $A = f^{-1}(a)$ *is the only nondegenerate inverse of a point under* f *and* $f \mid \dot{B}^n = 1$.

Proof. Let $\pi: B^n \to B^n/A$ be the projection map, and let $h: B^n/A \to B^n$ be a homeomorphism. Then $h\pi$ maps B^n into itself, and the only nondegenerate inverse of a point under $h\pi$ is $A = (h\pi)^{-1}(b)$. It is not difficult to show that $b \in \dot{B}^n$ and that $h\pi \mid \dot{B}^n$ is a homeomorphism of $\dot{B}^n$ onto itself. Let H be a homeomorphism of B^n such that $H \mid \dot{B}^n = h\pi \mid \dot{B}^n$. Then $f = H^{-1} h\pi$ is the required map.

LEMMA 2. *Suppose, in addition to the hypotheses of Lemma* 1, *we are given a neighborhood* U *of* A. *Then there exists a map* g *of* B^n *onto* B^n *such that* $A = g^{-1}(a)$ *is the only nondegenerate inverse of a point under* g *and* $g \mid \dot{B}^n \cup (B^n - U) = 1$.

Proof. Let f be the map provided by Lemma 1. Then f(U) is a neighborhood of the point a. Let Γ be a homeomorphism of B^n into f(U) such that $\Gamma \mid V = 1$, where V is a (small) neighborhood of a. Then

$$h = \begin{cases} f^{-1}\Gamma f & \text{on } B^n - A, \\ 1 & \text{on } A \end{cases}$$

is a well-defined homeomorphism of B^n into U. Let $Q^n = h(B^n)$. Then $A \subset Q^n \subset U$ and Q^n/A is homeomorphic to Q^n. This last assertion follows from the fact that hfh^{-1} maps Q^n onto itself and $A = (hfh^{-1})^{-1}(a)$ is the only nondegenerate inverse of a point under hfh^{-1}. Now apply Lemma 1 (with Q^n replacing B^n), and get a map g of Q^n onto Q^n such that $A = g^{-1}(a)$ is the only nondegenerate inverse of a point under g, and such that $g \mid \dot{Q}^n = 1$. Extend g by the identity map on $B^n - Q^n$. Note that $\dot{B}^n \cap \mathring{Q}^n = \emptyset$ (by invariance of domain), so that $g \mid \dot{B}^n = 1$.

LEMMA 3. *Let* $A \subset B^n \subset S^n$, *where* A *is cellular in* S^n, B^n *is a ball whose boundary is locally flat in* S^n, *and* $A \cap \dot{B}^n$ *is a point. Then* B^n/A *is homeomorphic to* B^n *if* $n \neq 3$.

Proof. Since A is cellular in S^n, S^n/A is homeomorphic to S^n, by [4]. In S^n/A, $\dot{B}^n$ is locally flat except possibly at the point $\langle A \rangle$. Hence, by [7], $\dot{B}^n$ is locally flat in S^n/A, and by [5], B^n/A is homeomorphic to B^n.

Proof of 4.6. Let $\alpha \subset S^n$ $(n \neq 3)$ be noncellular while each proper subarc of α is cellular. Suppose α pierces a locally flat sphere Σ^{n-1} at a point P. Let D be a fixed complementary domain of Σ^{n-1}. (D is an n-cell by [4], [5].) Let α_1, α_2 be the two subarcs of α determined by P, and suppose α_1 is the arc that is locally inside D, near P. (See Figure 1.) Using the collar [5] of Σ^{n-1} in D, construct a ball C^n with locally flat boundary $\dot{C}^n$ in S^n, so that $C^n \cap \alpha_2 = P$. Let B^n be the n-ball $S^n - \mathring{C}^n$. Then by Lemma 3, B^n/α_2 is homeomorphic to B^n. Let U be a neighborhood of α_2 in B^n such that $U \cap \alpha_1 = \emptyset$. (Note that the points of α_2 near P are not in B^n.) Then by Lemma 2 there exists a map g of B^n onto B such that $g \mid \dot{B}^n \cup (B^n - U) = 1$ and $\alpha_2 = g^{-1}(P)$ is the only nondegenerate inverse of a point under g. Extend g by the identity to $S^n - B^n$, to get a map g of S^n onto S^n such

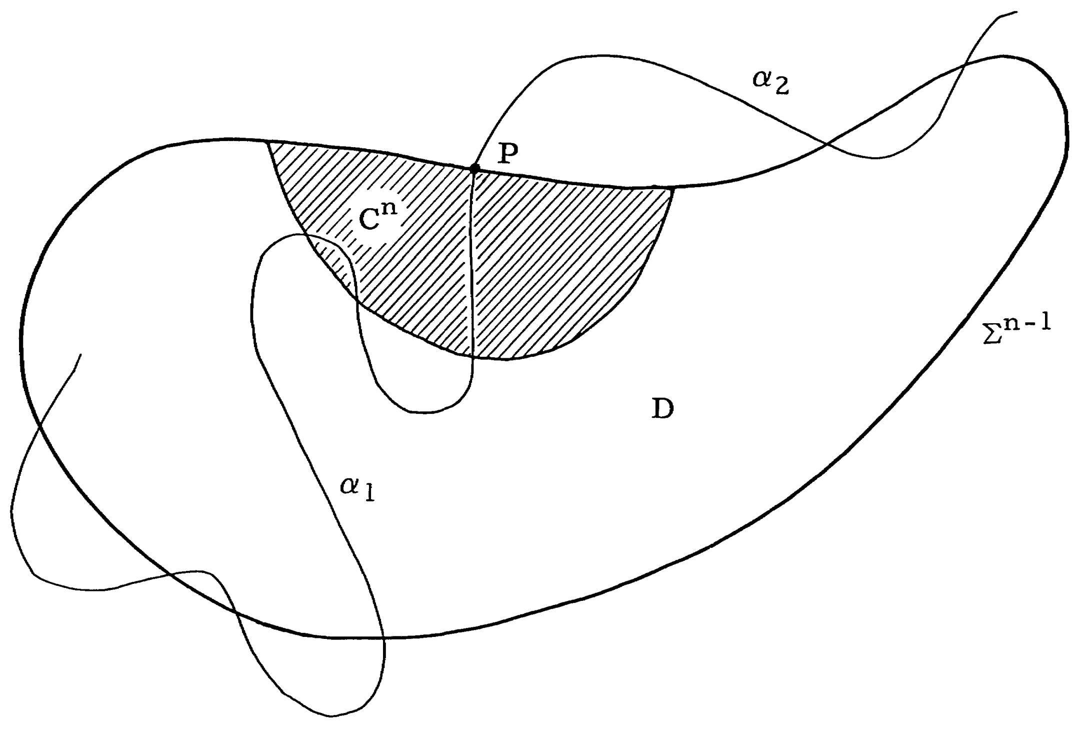

Figure 1.

that $\alpha_2 = g^{-1}(P)$ is the only nondegenerate inverse of a point under g and $g \mid \alpha_1 = 1$ (that $g \mid \alpha_1 = 1$ is the crucial fact that makes use of the piercing hypothesis). But by hypothesis, $\alpha_1 = g(\alpha)$ is cellular in S^n, and hence by [4] there exists a map f of S^n onto S^n such that α_1 is the only nondegenerate inverse of a point under f. Hence fg maps S^n onto S^n, and the only nondegenerate inverse of a point under fg is $g^{-1}(\alpha_1) = \alpha$. By [4], α is cellular in S^n, and this is a contradiction.

(4.7) *A wild simple closed curve in* S^n $(n > 3)$ *that has a cartesian-product neighborhood.*

This example is due to K. Kwun. Let α be a noncellular arc in the interior of a ball I^n $(n \geq 3)$. By [1], $(I^n/\alpha) \times R'$ is homeomorphic to $I^n \times R'$. Following [2] (or directly from [6]), one can prove that $(I^n/\alpha) \times S'$ is homeomorphic to $I^n \times S'$. Now form an $(n+1)$-sphere by attaching $(I^n/\alpha) \times S'$ to $S^{n-1} \times I^2$. The required simple closed curve is $J = \langle \alpha \rangle \times S'$. It has a trivial "normal bundle" in S whose fibre is I^n/α. J of course is wild, but in a completely homogeneous fashion.

(4.8) *A tame (and locally flat)* 2-*cell* I^2 *and an arc* α *in* S^4 *such that*

$$\alpha \cap I^2 = \mathring{\alpha} \cap \mathring{I}^2 = \mathring{I}^2 = 1 \textit{ point}$$

and such that every two-cell sufficiently close to I^2 *intersects* α.

The example is due to Zeeman [11, Chapter 6], and we mention it as an application of the construction α^*.

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