THE REGULAR CONVERGENCE THEOREM

C. N. Lee

Dedicated to R. L. Wilder on his seventieth birthday.

INTRODUCTION

The concept of regular convergence dates back to Whyburn [15], Wilder [14], and White [13]. A powerful theorem of E. E. Floyd [6, (2.3)], relating the Čech homology groups of compact Hausdorff spaces and their finite closed coverings, unifies several other theorems on regular convergence. The cohomology version of this theorem by Floyd was formulated and proved by E. Dyer [4, Theorem 1] by means of Leray's sheaf theory. This was the key theorem upon which Floyd based his proof of the major part of the conjecture by D. Montgomery that any compact Lie group acting on a compact manifold has only a finite number of conjugate classes of isotropy subgroups [6, Chapters 4 and 5].

The purpose of the present paper is to extend the theorem of Floyd to a larger class of spaces (see Theorems 3.3 and 3.6 below). As an application, we prove that the Borel-Moore homology groups with compact supports are naturally isomorphic with the singular homology groups on the class of locally compact Hausdorff HLC spaces (see Theorem 4.2).

The author is indebted to R. L. Wilder and F. Raymond for their encouragement of this work.

1. AXIOMATIC HOMOLOGY THEORY

We shall generally follow the notation and terminology of Eilenberg and Steenrod in [5]. Thus, a homology H_* is defined on an admissible category $\mathscr{A}$ so that it satisfies the seven axioms. Typical categories $\mathscr{A}$ with which we shall be concerned are $\mathscr{A}_H$, which consists of all Hausdorff pairs and all continuous maps between such pairs, and $\mathscr{W}$, which consists of pairs (X, A) such that X and A are of the homotopy type of a CW-complex and all continuous maps between such pairs. We adopt the convention that whenever we apply a functor to an object, the object is assumed to be in the category on which the functor is defined.

We shall consider some additional axioms for homology theory H_*.

Compact Support Axiom. If $z \in H_q(X, A)$, there exists a compact pair $(X', A') \subset (X, A)$ such that $z \in \operatorname{Im}(H_q(X', A') \to H_q(X, A))$.

A homology theory satisfying this axiom is called a *homology theory with compact supports*. Clearly, the singular homology theory over any coefficient group satisfies this axiom. Later in this section we shall consider another homology theory with compact support, namely the one considered by Borel and Moore in [3].

In [10], J. Milnor considered the following axiom for homology theory H_*.

Received September 8, 1966.
This work was supported in part by the National Science Foundation.

Additive Axiom. If a space X is the disjoint union of open subsets X_α of X, then the homomorphisms

$$H_q(X_\alpha) \to H_q(X)$$

induced by the inclusion maps $X_\alpha \subset X$ provide a representation of $H_q(X)$ as the direct sum of the $H_q(X_\alpha)$.

The dual statement for cohomology theory, with "direct sum" replaced by "direct product," is also called the additive axiom. A homology (or cohomology) theory satisfying the additive axiom is said to be additive. It is easy to see that any homology theory with compact supports is additive. Clearly, the singular homology theory is also additive. In fact, Milnor proved in [10] that any additive homology theory H_* is isomorphic with the singular homology theory (over the same coefficient group H_0(point)) on the category $\mathscr{W}$. Making use of this, we shall set up a natural transformation T from a singular homology theory (which we shall denote by H_*^s) to an arbitrary homology theory H_* with compact supports over the same coefficient group. For definiteness, we shall work in the category $\mathscr{A}_H$, and we assume that the homology theories are defined on $\mathscr{A}_H$. For any space Y, we shall denote by SY the singular complex over Y, by $|SY|$ its geometric realization given by J. B. Giever [7], and by $r: |SY| \to Y$ the natural projection maps. For each pair (X, A) in $\mathscr{A}_H$, we define a homomorphism

$$T_{X,A}: H_q^s(X, A) \to H_q(X, A)$$

as the composition of

$$H_q^s(X, A) \overset{r_*^{-1}}{\to} H_q^s(|SX|, |SA|) \to H_q(|SX|, |SA|) \overset{r_*}{\to} H_q(X, A),$$

where the unlabelled map is the natural isomorphism given by Milnor. Inasmuch as the map r and the Milnor equivalence are natural with respect to maps of pairs, so is the transformation $T: H_*^s \to H_*$. In particular, if $H_* \equiv H_*^s$, then T is an equivalence. We summarize these relations:

LEMMA 1.1. *There exists a natural transformation* T, *from the singular homology theory over a coefficient group to an arbitrarily given homology theory with compact supports over the same coefficient group, such that* T *is an equivalence on* $\mathscr{W}$.

The remainder of this section is devoted to a brief description of the relative form of the Borel-Moore homology theory with compact supports. First recall from [3] that Borel and Moore defined single space homology theory with compact supports on the category of locally compact Hausdorff spaces and proper maps. Following the suggestion by F. Raymond [11], we shall first define, for compact Hausdorff pairs (X, A), the Borel-Moore homology group $H_q^c(X, A)$ of the single space X - A with compact supports over any fixed coefficient group. The naturality of H_q^c with respect to maps is easily seen from the naturality of the Borel-Moore homology theory. Now, following the suggestion of Eilenberg and Steenrod [5, p. 255], we extend H_q^c to the category $\mathscr{A}_H$ of all Hausdorff pairs by

$$H_q^c(X, A) = \text{Dir Lim } H_q(X_\lambda, A_\lambda),$$

where the (X_λ, A_λ) are compact pairs contained in (X, A). The naturality of this extended H_q^c follows from that of the original H_q^c. Eilenberg and Steenrod [5, p. 225] suggest as an exercise the problem of showing that the extended H_*^c is also a homology theory on $\mathscr{A}_H$. Furthermore, the extended homology theory clearly satisfies the compact support axiom. We shall call this homology theory the *(relative)* Borel-Moore *homology theory with compact supports.*

2. HOMOLOGY SPECTRAL SEQUENCE OF AN OPEN COVERING

In order to state our theorem on homology spectral sequences, we need the dual concept of what Godement called a system of coefficients on a simplicial complex [6, pp. 42-44]. We shall call Godement's system a cohomology coefficient system, and we define its dual as follows.

A *homology coefficient system* $\mathscr{F}$ on a simplicial complex K consists of

(L-1) an abelian group $\mathscr{F}_\sigma$ for each simplex σ of K, and

(L-2) a homomorphism $\psi^\sigma_{\sigma'}: \mathscr{F}_\sigma \to \mathscr{F}_{\sigma'}$ for each pair σ', σ with $\sigma' \leq \sigma$ such that if $\sigma'' \leq \sigma' \leq \sigma$, then $\psi^\sigma_{\sigma''} = \psi^{\sigma'}_{\sigma''}\psi^\sigma_{\sigma'}$.

If we regard K as a category whose objects are simplicies and whose morphisms are incidence relations, then such a system $\mathscr{F}$ may be regarded just as a contravariant functor from K into the category $\mathscr{A}$b of abelian groups, and *vice versa.* In this sense, a cohomology coefficient system on K is a covariant functor from K (regarded as a category) into $\mathscr{A}$b.

For each orientation on K, we define the chain complex $C_*(K; \mathscr{F})$ to be the set of all functions $c: K \to \bigcup \mathscr{F}_\sigma$ such that $c(\sigma) \in \mathscr{F}_\sigma$ and $c(\sigma) = 0$ except for a finite number of σ. As usual, we require that c respect orientations of simplicies. Thus, if we denote by $g\sigma$ ($\sigma \in K$, $g \in \mathscr{F}_\sigma$) the element called an elementary chain of $C_*(K; \mathscr{F})$ (characterized by $(g\sigma)(\sigma) = g$ and $(g\sigma)(\tau) = 0$ for $\tau \neq \sigma$), then each $c \in C_*(K; \mathscr{F})$ is a finite sum of elementary chains. We define the boundary homomorphism $\partial: C_*(K; \mathscr{F}) \to C_*(K; \mathscr{F})$ by

$$\partial(g\sigma) = \sum_\tau \psi^\sigma_\tau([\sigma: \tau]g)\tau ,$$

where $[\sigma: \tau]$ is the incidence number. Consequently, the homology group $H_*(K; \mathscr{F})$ is defined to be the homology group of the chain complex $(C_*(K; \mathscr{F}), \partial)$. It is a standard exercise to show that $H_*(K; \mathscr{F})$ is independent of the orientation of K. Note also that there is the canonical way of grading $C_*(K; \mathscr{F})$, and hence $H_*(K; \mathscr{F})$, by means of the dimension of the simplicies. Finally, if K' is another simplicial complex and $\mathscr{F}'$ is a coefficient system for K', and if $f: K \to K'$ is a simplicial map and $F: \mathscr{F} \to \mathscr{F}'$ is a natural transformation compatible with f, then the induced homomorphism $H_*(K; \mathscr{F}) \to H_*(K'; \mathscr{F}')$ is well-defined in the obvious way.

Example 2.1. Let $\mathscr{F}$ be a covariant functor from the category of all open subsets of a space X and inclusion maps into the category $\mathscr{A}$b of abelian groups and homomorphisms. We shall assume that $\mathscr{F}$(empty set) = 0. If α is an open covering of X, then we define the local system (denoted by $\mathscr{F}^\alpha$) for the nerve X_α by the formula $F_\sigma = F(\mathrm{car}_\alpha(\sigma))$; here

$$\mathrm{car}_\alpha(\sigma) = u_{i_0} \cap \cdots \cap u_{i_q},$$

where $\{u_{i_0}, \cdots, u_{i_q}\}$ is the set of all vertices of $\sigma \in X_\alpha$; if $\tau \leq \sigma$, then

$$\psi_\tau^\sigma\colon F(\mathrm{car}_\alpha(\sigma)) \to F(\mathrm{car}_\alpha(\tau))$$

is the homomorphism induced by the inclusion map $\mathrm{car}_\alpha(\sigma) \subset \mathrm{car}_\alpha(\tau)$. We denote the homology group of X_α over such a coefficient system by $H_*(X_\alpha; \mathscr{F}^\alpha)$. If β is another open covering of X and β refines α, then there exists a unique homomorphism $\pi_*\colon H_*(X_\beta; \mathscr{F}^\beta) \to H_*(X_\alpha; \mathscr{F}^\alpha)$ induced by a projection map $\pi\colon \beta \to \alpha$. The cohomology form of all these is well known (see Godement [6, Chapter II, Section 5], for example).

THEOREM 2.2. *Let* H_* *be a homology theory with compact supports defined on* $\mathscr{A}_H$. *If* α *is a locally finite open covering of a normal space* X, *there exists a homology spectral sequence* $E(\alpha) = \{E^r_{p,q}\}$ *such that*

$$E^2_{p,q} \approx H_p(X_\alpha; \mathscr{H}^\alpha_q),$$

where $\mathscr{H}^\alpha_q$ *is the coefficient system defined by* $\mathscr{H}_q(\sigma) = H_q(\mathrm{car}_\alpha(\sigma))$ (see Example 2.1 above), *and* E^∞_n *is the graded group of a suitable filtration of* $H_n(X)$. *Furthermore, if* $f\colon X \to Y$ *is a map between the normal spaces, and if* α *and* β *are locally finite open coverings of* X *and* Y, *respectively, such that* $f(\alpha)$ *refines* β, *then there is an induced homomorphism* $E(\alpha) \to E(\beta)$ *compatible with* $f_*\colon H_n(X) \to H_n(Y)$.

Since we shall publish a more general result in a forthcoming joint paper with F. Raymond, we shall state the definition of such a spectral sequence, but only outline the proof of the conclusions.

Following E. Dyer in [4], we consider the subspace $\mathscr{X}_\alpha = \bigcup \mathrm{car}_\alpha(\sigma) \times |\sigma|$ of the product space $X \times |X_\alpha|$, where the union is taken over the set of all simplices $\sigma \in X_\alpha$. It is not difficult to see that the projection map $\mathscr{X}_\alpha \to X$ is a homotopy equivalence, provided X is normal and α is locally finite. In fact, we take the canonical map $\phi_\alpha\colon X \to |X_\alpha|$, defined by means of a partition of unity; then the map $X \xrightarrow{\psi_\alpha} \mathscr{X}_\alpha$ defined by $\psi_\alpha(x) = (x, \phi_\alpha(x))$ is a homotopy inverse of the projection map $\mathscr{X}_\alpha \to X$. Furthermore, the projection map $\mathscr{X}_\alpha \to |X_\alpha|$ composed with ψ_α is identical with ϕ_α. The construction $\mathscr{X}_\alpha$ is natural with respect to maps $X \to Y$ and the compatible projection maps between the open coverings.

The desired spectral sequence is now obtained from the exact couple (as defined by Hu [9, p. 234]) that is associated with the filtration

$$\emptyset = \mathscr{X}^{-1}_\alpha \subset \mathscr{X}^0_\alpha \subset \mathscr{X}^1_\alpha \subset \cdots,$$

of $\mathscr{X}_\alpha$, where $\mathscr{X}^p_\alpha = \pi^{-1}|X^p_\alpha|$, where $\pi\colon \mathscr{X}_\alpha \to |X_\alpha|$ is the projection map, and where $|X^p_\alpha|$ is the p-skeleton of the polyhedron of the nerve X_α. In the forthcoming paper mentioned earlier, we shall show the computation of the E^2_{pq} to confirm the conclusion. Furthermore, using the notation of Hu [9], we see that

$$E^\infty_{p,q} \approx \overline{D}_{p,q}/\overline{D}_{p-1,q+1} \quad \text{and} \quad \operatorname*{Dir\,Lim}_{p\to\infty} H_n(\mathscr{X}^p_\alpha) \xrightarrow{\approx} H_n(\mathscr{X}_\alpha).$$

This is how we get the conclusion about E^∞_n stated in the theorem. The last statement about the naturality of the spectral sequence is the consequence of the naturality of the construction of $\mathscr{X}_\alpha$. This completes the outline of the proof.

REMARK 2.3. *If we assume that* $H_q(U) = 0$ *for* $q < 0$ *and every open subset* U *of* X, *then the composition of the homomorphisms*

$$H_n(X) \approx \overline{D}_{n,0} \xrightarrow{\text{epi}} E^{\infty}_{n,0} \xrightarrow{\text{mono}} E^2_{n,0} \approx H_n(X_\alpha; \mathscr{H}_0) \to H_n(X_\alpha)$$

is equivalent to the homomorphism $H_n(X) \to H_n(X_\alpha)$ *induced by the canonical map* $\phi_\alpha: X \to X_\alpha$.

Proof. Take the filtration of $\mathscr{X}_\alpha$ as before, and that of $|X_\alpha|$ by its skeletons. Then the projection map $\pi: \mathscr{X}_\alpha \to |X_\alpha|$ preserves the filtration, and hence it induces a homomorphism of the spectral sequence of $\mathscr{X}_\alpha$ into that of $|X_\alpha|$. Now the conclusion follows, since the spectral sequence of $|X_\alpha|$ collapses.

REMARK 2.4. *The cohomology spectral sequence of a locally finite open covering can be obtained in the same way as for homology in Theorem* 2.2 *provided that the cohomology theory used satisfies the additive axiom.*

REMARK 2.5. *We could also obtain Theorem* 2.2 *for a locally finite closed covering of* X *instead of an open covering, provided that the interiors of the sets of the closed covering make a covering of the space* X. *(This latter requirement ensures that the corresponding projection map* $\mathscr{X}_\alpha \to X$ *is still a homotopy equivalence.) We can say the same for cohomology.*

3. THE REGULAR CONVERGENCE THEOREM

In this section, we suppose that H_* is a fixed homology theory with compact supports defined on $\mathscr{A}_H$.

A map f: $X \to Y$ is said to be *homologically locally* n*-trivial with respect to* H_* if for each $x \in X$ and an open neighborhood u of $y = f(x)$ there exists an open neighborhood v of x such that $f(v) \subset u$ and the induced homomorphism $\tilde{H}_q(v) \to \tilde{H}_q(u)$ is trivial for all $q \leq n$, where

$$\tilde{H}_q(Z) = \operatorname{Ker}(H_q(Z) \to H_q(\text{point}))$$

for any space Z. The concept of a *cohomologically locally* n*-trivial map* is similarly defined for a given cohomology theory.

A special case of the following lemma was proved by Floyd [4, (2.2)] for finite closed coverings.

LEMMA 3.1. *Let* X *and* Y *be paracompact Hausdorff spaces. If* f: $X \to Y$ *is homologically locally* n*-trivial, then for each open covering* β *of* Y *there exist a locally finite open covering* α *of* X *and a projection map* $\pi: \alpha \to \beta$ *with* $f(u) \subset \pi(u)$ $(u \in \alpha)$ *such that the induced homomorphism*

$$f_*: H_q(u_{i_0} \cap \cdots \cap u_{i_p}) \to H_q(\pi u_{i_0} \cap \cdots \cap \pi u_{i_p})$$

is trivial for $q \leq n$.

The lemma can be proved by an easy modification of the argument given by Floyd [4].

The cohomology version of Lemma 3.1 is easily formulated and similarly proved.

REMARK 3.2. *Let* f: $X \to Y$ *and* $\pi: \alpha \to \beta$ *satisfy the conditions in the preceeding lemma, with* $n = 0$. *Then the coefficient system* $\operatorname{Im}(\mathscr{H}_0^\alpha \xrightarrow{\pi_*} \mathscr{H}_0^\beta)$ *over the nerve* Y_β *is isomorphic with the simple system* $H_0(\text{point})$.

This follows immediately from the definitions of the reduced homology group $\tilde{H}_*$, the homology local 0-triviality, and the commutative diagram

$$\begin{array}{ccc} H_0(u_{i_0} \cap \cdots \cap u_{i_p}) & \rightarrow & H_0(\text{point}) \\ \downarrow f_* & & \downarrow \approx \\ H_0(\pi u_{i_0} \cap \cdots \cap \pi u_{i_p}) & \rightarrow & H_0(\text{point}). \end{array}$$

THEOREM 3.3 *(Regular Convergence). Assume* $H_q \equiv 0$ *for* $q < 0$. *Let* $f: X \rightarrow Y$ *be the composition of* $n + 1$ *maps*

$$X = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_n \xrightarrow{f_n} X_{n+1} = Y$$

such that each map is homologically locally n*-trivial and each* X_m *is a paracompact Hausdorff space. Then, for each open covering* β *of* Y, *there exist a locally finite open covering* α *of* X *and a projection map* $\pi: \alpha \rightarrow \beta$ *with* $f(u) \subset \pi(u)$ $(u \in \alpha)$ *such that in the commutative diagram*

$$\begin{array}{ccc} H_p(X) & \xrightarrow{\phi_{\alpha*}} & H_p(X_\alpha) \\ \downarrow f_* & & \downarrow \pi_* \\ H_p(Y) & \xrightarrow{\phi_{\beta*}} & H_p(Y_\beta) \end{array}$$

we have the inclusion relations

(1) $\text{Ker } \phi_{\alpha*} \subset \text{Ker } f_*$ *for* $p \leq n$,

(2) $\text{Im } \pi_* \subset \text{Im } \phi_{\beta*}$ for $p \leq n + 1$.

Proof. We shall follow the general idea of Dyer in his proof of the corresponding theorem [3, Theorem 1]. By repeated application of Lemma 3.1, we find for each j a locally finite open covering α_j of X_j and a projection map $\pi_j: \alpha_j \rightarrow \alpha_{j+1}$ with $f_j(u) \subset \pi_j(u)$ $(u \in \alpha_j)$ that induces the trivial homomorphism

$$\tilde{H}_p(u_{i_0} \cap \cdots \cap u_{i_q}) \rightarrow \tilde{H}_p(\pi_j u_{i_0} \cap \cdots \cap \pi_j u_{i_q})$$

for $p \leq n$ and all q. Here we assume without loss of generality that $\alpha_{n+1} = \beta$.

Let $E(\alpha_j) = \{{}_jE^r_{p,q}\}$ be the spectral sequence of the covering α_j, and let $\theta_j: E(\alpha_j) \rightarrow E(\alpha_{j+1})$ be the induced homomorphism for each $j \leq n$. We know by the definition of α_j that ${}_jE^r_{p,q} \rightarrow {}_{j+1}E^r_{p,q}$ is the trivial homomorphism for $1 \leq q \leq n$, $1 \leq r \leq \infty$, and all p. Denote by ${}_j\overline{D}_{p,q}$ the $\overline{D}_{p,q}$-term for $E(\alpha_j)$. Since ${}_j\overline{D}_{p,q} = 0$ for $p < 0$, $H_n(X_j) = {}_j\overline{D}_{0,n}$.

To prove (1), it is clearly sufficient to consider only the case $p = n$. Let $a \in H_n(X)$ with $\phi_{\alpha*}(a) = 0$. We need only show that $(f_0 \cdots f_k)_*(a) \in {}_{k+1}\overline{D}_{n-k-1,k+1}$ for each k, since then the particular case $k = n$ reduces to the relation

$$(f_0 \cdots f_n)_*(a) \in {}_{n+1}\overline{D}_{-1,n+1} = 0$$

and hence $a \in \operatorname{Ker} f_*$. We shall do this by an induction on k. Consider the commutative diagram

$$\begin{array}{ccccc} H_n(X) & \xrightarrow{\text{epi}} & {}_0E^{\infty}_{n,0} & \xrightarrow{\text{mono}} & {}_0E^2_{n,0} \\ \downarrow f_{0*} & & \downarrow & & \downarrow \\ H_n(X_1) & \xrightarrow{\text{epi}} & {}_1E^{\infty}_{n,0} & \xrightarrow{\text{mono}} & {}_1E^2_{n,0}. \end{array}$$

From Lemma 3.2 it follows that the image of a under the homomorphism composed of the two horizontal maps in the upper line, followed by the vertical map on the far right, is zero. Hence the image of $f_{0*}(a)$ under the homomorphism $H_n(X_1) \to {}_1E^{\infty}_{n,0}$ is zero. This implies that $f_{0*}(a) \in {}_1\overline{D}_{n-1,1}$. Now let $k > 0$, and assume that we have already proved $(f_0 \cdots f_k)_*(a) \in {}_{k+1}\overline{D}_{n-k-1,k+1}$ for smaller k. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_j\overline{D}_{p-1,q+1} & \xrightarrow{\text{incl}} & {}_j\overline{D}_{p,q} & \xrightarrow{\text{proj}} & {}_jE^{\infty}_{p,q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_{j+1}\overline{D}_{p-1,q+1} & \xrightarrow{\text{incl}} & {}_{j+1}\overline{D}_{p,q} & \xrightarrow{\text{proj}} & {}_{j+1}E^{\infty}_{p,q} & \longrightarrow & 0. \end{array}$$

We complete the induction by inspection of this diagram, for $j = k + 1$, $p = k + 2$, and $n = p + q$.

In order to prove (2), it is sufficient to consider the case $p = n + 1$. Let $a \in H_{n+1}(X_\alpha)$. We must show that $\phi_{\beta*}(b) = \pi_*(a)$ for some $b \in H_{n+1}(Y)$. From Lemma 3.2 it follows that there is a $b_1 \in {}_1E^2_{n+1,0}$ such that $\bar{\phi}_{1*}(b_1) = \pi_{0*}(a)$, where $\bar{\phi}_{1*}\colon {}_1E^2_{n+1,0} \to H_{n+1}(X_{\alpha_1})$. Now, we need only show $\theta_k \cdots \theta_1(b_1) = b_k$ represents an element of ${}_kE^{k+1}_{n+1,0}$ for $k = 1, \cdots, n+1$, since then b_{n+1} represents an element of ${}_{n+1}E^{n+2}_{n+1,0} = {}_{n+1}E^{\infty}_{n+1,0} = H_{n+1}(Y)$. Consider the commutative diagram

$$\begin{array}{ccc} {}_1E^2_{n+1,0} & \xrightarrow{d_2} & {}_1E^2_{n-1,0} \\ \downarrow \theta_1 & & \downarrow \theta_1 \\ {}_2E^2_{n+1,0} & \xrightarrow{d_2} & {}_2E^2_{n-1,1}. \end{array}$$

Since θ_1 is trivial on ${}_1E^2_{n-1,0}$, we see that $d_2\,\theta_1(b_1) = 0$, and this implies that $\theta_1(b_1) = b_2$ represents an element of ${}_2E^3_{n+1,0}$. Repeating a similar argument, we get the conclusion; this completes the proof of Theorem 3.3.

REMARK 3.4. *In Theorem* 3.3, *we made the unfortunate assumption that* $H_q \equiv 0$ *for* $q < 0$; *this excludes the example of the* Borel-Moore *homology theory with compact supports. Denote this latter homology theory by* H^c_*. *Then* $H^c_q \equiv 0$ *for* $q < -1$, *and we can prove the same theorem for* H^c_*, *where the number of factor maps must be increased by one.* The proof, although more complicated, involves no new essential difficulty, and we leave it to the reader.

Assume that $H_q \equiv 0$ for $q < -1$.

COROLLARY 3.5. *Let* X *be a locally compact Hausdorff space that is homologically locally* n-*connected; in other words, let the identity map* $1_X\colon X \to X$ *be homologically locally* n-*trivial. If* A *and* B *are compact subsets of* X *with* $A \subset \operatorname{Int} B$, *and if* β *is a finite open covering of* B, *then there exists a finite open covering* α *that refines* β, *and such that in the diagram*

$$\begin{array}{ccc} H_q(A) & \xrightarrow{\phi_{\alpha *}} & H_q(A_\alpha) \\ \downarrow i_* & & \downarrow \pi_* \\ H_q(B) & \xrightarrow{\phi_{\beta *}} & H_q(B_\beta) \end{array}$$

we have the inclusion relations

(1) $\operatorname{Ker} \phi_{\alpha *} \subset \operatorname{Ker} i_*$ for $q \leq n$,

(2) $\operatorname{Im} \pi_* \subset \operatorname{Im} \phi_{\beta *}$ for $q \leq n+1$.

Proof. Simply take $n+1$ intermediate compact subsets

$$A = A_{-1} \subset A_0 \subset \cdots \subset A_n \subset A_{n+1} = B$$

such that $A_i \subset \operatorname{Int} A_{i+1}$ for all i. Clearly, the inclusion map $A_i \subset A_{i+1}$ is n-trivial. Now the conclusion follows from Theorem 3.3 and the subsequent remark.

For completeness, we shall record without proof the cohomology version of Theorem 3.3 that was first formulated by Dyer [4, Theorem 1]. Let H^* be an additive axiomatic cohomology theory defined on $\mathscr{A}_H$ (see Section 1).

THEOREM 3.6. *Assume* $H^q \equiv 0$ *for* $q < 0$. *Let* $f\colon X \to Y$ *be the composition of* $n+1$ *maps each of which is cohomologically locally* n-*trivial, and suppose the spaces involved are paracompact Hausdorff spaces. Then, for each open covering* β *of* Y, *there exist a locally finite open covering* α *of* X *and a projection map* $\pi\colon \beta \to \alpha$ *with* $f(u) \subset \pi(u)$ $(u \in \alpha)$ *such that in the diagram*

$$\begin{array}{ccc} H^p(X) & \xleftarrow{\phi_\alpha^*} & H^p(X_\alpha) \\ \uparrow f^* & & \uparrow \pi^* \\ H^p(Y) & \xleftarrow{\phi_\beta^*} & H^p(Y_\beta) \end{array}$$

we have the inclusion relation

(1) $\operatorname{Ker} \phi_\beta^* \subset \operatorname{Ker} \pi^*$ *for* $p \leq n+1$,

(2) $\operatorname{Im} f^* \subset \operatorname{Im} \phi_\alpha^*$ *for* $p \leq n$.

REMARK 3.7. *Theorems* 3.3 *and* 3.6 *can also be proved for locally finite closed coverings, provided the interiors of the sets of each covering constitute an open covering.*

4. APPLICATION

We shall say that a space X is HLC^n with respect to a homology theory H_* if the identity map of X onto itself is homologically locally n-trivial with respect to H_*. HLC^n is the homology version of CLC^n. Denote by H_*^s the singular homology theory over a fixed coefficient group R, and by H_*^c the Borel-Moore homology theory with compact supports over the same coefficient group R. HLC^n with respect to H_*^s with R = Z (the group of integers) is merely called HLC^n, and HLC^n with respect to H_*^c with R = Z is called hlc^n.

LEMMA 4.1. *Let* X *be a locally compact Hausdorff space. If* X *is* HLC^n, *then* X *is* hlc^{n-1}.

This is easy to prove, and hence we leave it to the reader.

Now, recall from Section 1 the natural transformation $T: H_*^s \to H_*^c$, which is an equivalence on the category $\mathcal{W}$. The following theorem implies that T is also an equivalence on HLC spaces.

THEOREM 4.2. *Let* (X, A) *be a Hausdorff pair. If* X *and* A *are of the homotopy type of a locally compact Hausdorff space that is* HLC^n, *then*

$$T_{X,A}: H_q^s(X, A) \to H_q^c(X, A)$$

is an isomorphism for $q \leq n - 1$, *and it is surjective for* $q = n$.

Proof. We need only prove the theorem for $A = \emptyset$, since the relative form stated above will then follow from the Five Lemma. It is also sufficient to consider the case where X is actually a locally compact Hausdorff space and is HLC^n.

We shall encounter the diagram

$$\begin{array}{ccccccc}
H_q^s(A) & \to & H_q^s(A_\alpha) & \underset{\approx}{\overset{T}{\to}} & H_q^c(A_\alpha) & \leftarrow & H_q^c(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_q^s(B) & \to & H_q^s(B_\beta) & \underset{\approx}{\overset{T}{\to}} & H_q^c(B_\beta) & \leftarrow & H_q^c(B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_q^s(C) & \to & H_q^s(C_\gamma) & \underset{\approx}{\overset{T}{\to}} & H_q^c(C_\gamma) & \leftarrow & H_q^c(C),
\end{array}$$

where $A \subset B \subset C$ are compact subsets of X and $\alpha > \beta > \gamma$ are their respective finite open coverings.

To see the injectivity of T_X on $H_q^s(X)$, let $a \in H_q^s(X)$, with $T_X(a) = 0$. By the compact support axiom, there exist a compact subset A of X and an element $a' \in H_q^s(X)$ such that $a = i_*(a')$, where $i: A \subset X$. Since

$$0 = T_A(a) = T_X i_*(a') = i_* T_A(a') \in H_q^c(A),$$

there is a compact subset B with $A \subset B$ such that $j_* T_A(a') = 0$, where $j: A \subset B$. This follows from the general fact that

$$\text{Dir Lim } H_q^c(C) \approx H_q^c(X),$$

where the limit is taken over the set of all compact subsets C of X; this is true for any homology theory with compact supports (see Spanier [12, p. 204]). We take another compact subset C with $B \subset \operatorname{Int} C$, and by applying Corollary 3.4 we can find finite open coverings $\alpha > \beta > \gamma$ of A, B, and C, respectively, such that

$$\operatorname{Ker}(H_q^s(B) \to H_q^s(B_\beta)) \subset \operatorname{Ker}(H_q^s(B) \to H_q^s(C)) \qquad (q \leq n).$$

It follows by easy diagram-tracing on the above diagram that a' is mapped onto 0, under the homomorphism $H_q^s(A) \to H_q^s(C)$. Hence $a = 0$, and this completes the proof of the injectivity of T_X, for $q \leq n$.

To see the surjectivity of T_X, let $a \in H_q^c(X)$ $(q \leq n - 1)$. By the compact support axiom, there exist a compact subset A and $a' \in H_q^c(A)$ such that $i_*(a') = a$, where $i: A \subset X$. Choose any compact subsets B and C such that $A \subset \operatorname{Int} B$ and $B \subset \operatorname{Int} C$. Since X is hlc^{n-1} (by Lemma 4.1), we can apply the regular convergence theorem, and get finite open coverings $\alpha > \beta > \gamma$ for A, B, and C, respectively, such that

$$\operatorname{Im}(H_q^s(A_\alpha) \to H_q^s(B_\beta)) \subset \operatorname{Im}(H_q^s(B) \to H_q^s(B_\beta)),$$

$$\operatorname{Ker}(H_q^c(B) \to H_q^c(B_\beta)) \subset \operatorname{Ker}(H_q^c(B) \to H_q^c(C))$$

for $q \leq n - 1$. Easy diagram-tracing now shows that there exists an element $a'' \in H_q^s(C)$ with $T_C(a'') = j_*(a')$, where $j: A \subset C$. This completes the proof.

REFERENCES

1. A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies 46, Princeton, 1960.

2. A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France 89 (1961), 461-513.

3. A. Borel and J. C. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. 7 (1960), 137-159.

4. Eldon Dyer, *Regular mappings and dimension*, Ann. of Math. (2) 67 (1958), 119-149.

5. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.

6. E. E. Floyd, *Closed coverings in Čech homology theory*, Trans. Amer. Math. Soc. 84 (1957), 319-337.

7. J. B. Giever, *On the equivalence of two singular homology theories*, Ann. of Math. (2) 51 (1950), 178-191.

8. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Ind. 1252, Hermann, Paris, 1958.

9. S. T. Hu, *Homotopy Theory*, Academic Press, New York and London, 1959.

10. J. Milnor, *On axiomatic homology theory*, Pacific J. Math. 12 (1962), 337-341.

11. F. Raymond, *Some remarks on the coefficients used in the theory of homology manifolds*, Pacific J. Math. 15 (1965), 1365-1376.

12. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

13. P. A. White, *Regular convergence*, Bull. Amer. Math. Soc. 60 (1954), 431-443.

14. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32, 1949.

15. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28, 1942.

The University of Michigan