CONCORDANCE OF DIFFERENTIABLE STRUCTURES–TWO APPROACHES

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Dedicated to Professor R. L. Wilder on his seventieth birthday.

The following two basic problems in differential topology have attracted considerable attention in the last few years.

I. Given a piecewise-linear manifold K, find for it a compatible differentiable structure α.

II. Classify such structures, up to diffeomorphism or some other suitable equivalence relation.

A *piecewise-linear manifold* is a complex K that is locally piecewise-linearly homeomorphic to euclidean space R^n. *Compatibility* means that for some subdivision of K, each simplex σ of the subdivision inherits its usual differentiable structure; we restrict ourselves to structures that are compatible.

There are several possible equivalence relations one might study; the one of particular interest to us is that of concordance: Two differentiable structures α and β on K are said to be *concordant* if there exists a differentiable structure γ on $K \times I$ that equals α on $K \times 0$ and β on $K \times 1$. (The structure γ is called a *concordance* between α and β; it is a *strong concordance* if each level manifold $K \times t$ is a differentiable submanifold of $(K \times I)_\gamma$.) Concordance is a natural equivalence relation, in the sense that it establishes a connection between our two problems—constructing a concordance is simply the problem of finding a differentiable structure on $K \times I$ that extends a preassigned differentiable structure on the boundary. The question of the relation of concordance to diffeomorphism we leave in abeyance for the moment.

Our purpose is to give a brief survey of the subject, to outline the two main attacks that have been made on these problems, to indicate the connections between them, and to suggest some promising directions for future investigation.

1. THE GEOMETRIC APPROACH

The first of these attacks is the geometric one we made a few years ago on Problem II [17]. It involves the groups Γ_n, defined earlier by Thom in his own attacks (not completely successful) on these problems. Γ_n is defined as the group of diffeomorphisms of S^{n-1}, modulo the subgroup consisting of those extendable to diffeomorphisms of the ball B^n; it is abelian, and it has been proved to vanish for $n \leq 4$ [2], [16], [22].

Our general approach is the following: We attempt to classify differentiable structures on K — up to diffeomorphism for the moment. Our way of proceeding is to take a fixed structure α on K, and any other structure β, and to try to construct a diffeomorphism of K_α with K_β. The obstructions that occur provide some

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measure at least of the number of distinct differentiable structures admitted by K.

More specifically, we begin with the identity map $f\colon K_\alpha \to K_\beta$ and seek to modify it so that it becomes a diffeomorphism. Passing to a subdivision of K, if necessary, we may assume that each simplex of K inherits its usual differentiable structure under both α and β. Then f is a diffeomorphism on all but the (n - 1)-skeleton of K; the problem is to make it a diffeomorphism everywhere.

It is more convenient to consider the dual cell decomposition of K rather than its simplicial decomposition. The homeomorphism f is a diffeomorphism in a neighborhood of the dual 0-skeleton of K. At the general step of our construction, we assume we have a homeomorphism f' that is a diffeomorphism in a neighborhood of the dual (p - 1)-skeleton of K, and we seek to alter it to a diffeomorphism in a neighborhood of the dual p-skeleton. For descriptive purposes, we imagine the general dual p-cell c_p to be a smooth ball in K_α that f' maps onto itself. Now $f' | \mathrm{Bd}\, c_p$ defines a diffeomorphism of the (p - 1)-sphere with itself; if this diffeomorphism is extendable to a diffeomorphism of the ball c_p, then it seems likely (and is in fact the case) that f' may be altered so as to be a diffeomorphism in a neighborhood of c_p. We define the obstruction cochain $\lambda^p f'$ as the function that assigns to each dual cell c_p the element of Γ_p represented by $f' | \mathrm{Bd}\, c_p$.

Now this obstruction, which arises in the process of "smoothing" f to a diffeomorphism, may depend on the various choices made in the passage from f to f'. Upon examining the effect of these choices, we find the following: Altering the choice one step back alters $\lambda^p f'$, which is a cocycle, within its class in the cohomology group $H^p(K; \Gamma_p)$. Altering two steps back alters this cohomology class by an element in the image of a certain homomorphism

$$\Lambda^2\colon H^{p-2}(K; \Gamma_{p-1}) \to H^p(K; \Gamma_p).$$

Altering three steps back alters the resulting class by an element in the image of a homomorphism

$$\Lambda^3\colon H^{p-3}(K; \Gamma_{p-2}) \cap (\text{kernel}\, \Lambda^2) \to H^p(K; \Gamma_p)/(\text{image}\, \Lambda^2),$$

and so on. The class of $\lambda^p f'$ in the group

$$(\cdots((H^p(K; \Gamma_p)/\text{image}\, \Lambda^2)/\text{image}\, \Lambda^3)/\cdots)/\text{image}\, \Lambda^p$$

depends only on f; we denote it by $o^p(f)$, and we call it the *obstruction* in dimension p *to smoothing* f. The map f may be smoothed to a diffeomorphism if and only if every obstruction $o^p(f)$ vanishes.

We have not yet done what we set out to do; for $o^p(f)$ is not the obstruction to constructing a diffeomorphism between K_α and K_β, but rather the obstruction to constructing a diffeomorphism in a specific way--as a smoothing of the identity map f. We have a new equivalence relation between differentiable structures: two are equivalent if the identity map f may be smoothed to a diffeomorphism. We need to clarify the connection between this relation and the one we have promised to discuss—that of concordance. The answer is that they are the same.

In one direction, this follows as a corollary of the obstruction theory just outlined; it is embodied in the following theorem.

I-COBORDISM THEOREM [18]. *If γ is a concordance between α and β, then there exists a diffeomorphism*

$$g\colon (K \times I)_\gamma \to K_\beta \times I.$$

(Much stronger results than this hold when K is compact and $n \geq 5$; see the h-cobordism theorem of Smale [15], [23] and the s-cobordism theorem of Barden, Mazur, and Stallings [6], [11], [12].)

For the proof, we merely note that the identity map $f\colon (K \times I)_\gamma \to K_\beta \times I$ is already a diffeomorphism when restricted to $K \times 1$, so that the obstructions to smoothing f lie in the cohomology of $K \times I$ modulo $K \times 1$, which vanishes. Hence the diffeomorphism g not only exists, but may be chosen as a smoothing of the identity map. Restricting attention to the bottom face, we have the following result:

COROLLARY. *If α is concordant to β, then the identity map $K_\alpha \to K_\beta$ may be smoothed to a diffeomorphism.*

This shows in particular that concordance implies diffeomorphism. Concordance is in fact strictly stronger than diffeomorphism, as such a simple example as $S^i \times R^j$ shows [4], [20]. $S^i \times S^j$ is another example.

The proof in the other direction requires a careful geometrical argument, which we have but recently carried out; in [21] we prove the following:

STRONG CONCORDANCE THEOREM. *If the identity map $K_\alpha \to K_\beta$ may be smoothed to a diffeomorphism, then α is strongly concordant to β.*

These two theorems show that our classes $o^p(f)$ are actually obstructions to the existence of a concordance between α and β. As a consequence, we can get an upper bound on the number of distinct concordance classes that K may have. If C(K) denotes the set of concordance classes of differentiable structures on K, then

$$\text{order } C(K) \leq \text{order} \sum_p (H^p(K;\ \Gamma_p)/\text{image } \Lambda^2/\text{image } \Lambda^3/\cdots).$$

Questions about possible equality reduce to questions about the realizability of certain cohomology classes as obstructions. We can answer these questions in some cases.

For example,

$$\text{order } C(S^n) = \text{order } \Gamma_n .$$

The computation in this case is particularly simple, for there is only one nonvanishing group, and all possible obstructions are realizable. The facts here are stronger than this, however. For $C(S^n)$ is a group if we use the connected sum of manifolds [7] as the operation. And there is a monomorphism $\Gamma_n \to C(S^n)$ defined by assigning to each element of Γ_n (represented by an orientation-preserving diffeomorphism ϕ of S^{n-1}) the differentiable structure on S^n that we obtain by pasting two balls together along their boundaries by the map ϕ. Our computation shows that this monomorphism is an isomorphism.

We should note here that for $n \geq 5$, Smale has proved something far stronger than this [15], [23]: not only is Γ_n isomorphic to $C(S^n)$, but it is also isomorphic to the group Θ_n of h-cobordism classes of homotopy spheres. The importance of this result lies in the fact that a good deal of information about Θ_n is known; Milnor and Kervaire have computed it in low dimensions (for example, $\Theta_5 = \Theta_6 = 0$ and $\Theta_7 \cong Z_{28}$), and they have shown it to be finite for $n \neq 3$ [7].

A second example is the following:

$$\text{order } C(S^i \times S^j) = \text{order } \Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j}.$$

This case is also simple. One proves easily that all possible obstructions are realizable, and that all the Λ^i are zero.

2. OTHER APPROACHES

There have been other geometric approaches to these problems, of varying degrees of completeness and success. One of the first was that of Thom, whose work we have already mentioned [24]. The difficulty with his approach was that he needed to assume the asphericity of certain spaces L_n in order to construct his obstruction theory; this assumption has since been proved to be false [8]. The space L_n has appeared elsewhere in differential topology; it is the space of piecewise-linear homeomorphisms of the standard n-simplex Δ^n with itself that equal the identity on the boundary. It is topologized as follows: Let L_δ be the subspace of L_n consisting of all maps that are linear with respect to the subdivision Δ_δ of Δ; give L_δ the compact open topology; and let a set be closed in L_n if it is a closed subset of some L_δ. Asphericity of L_n then means that every map $S^k \to L_\delta$ is homotopic to a constant in L_ε, for some subdivision Δ_ε of Δ_δ.

J. H. C. Whitehead has also attacked these two problems [26]. Because his work also involved the asphericity of the spaces L_n (known to hold only for small n), his results were of importance primarily for 3- and 4-dimensional manifolds.

In [8], N. Kuiper studied Problem II using a different equivalence relation, that of *homotopy* of differentiable structures. In the proof of the strong concordance theorem mentioned above, we show that if α and β are concordant, then the concordance γ between them may be assumed to be induced by a level-preserving piecewise-smooth triangulation

$$g: K \times I \to K_\beta \times I.$$

If it should happen that g is smooth on each cell $\sigma \times I$ for each closed simplex σ of some subdivision K' of K, then γ is called a *homotopy* between α and β.

In [19], we constructed an obstruction theory for Problem I, that of imposing a differentiable structure on a manifold. The obstructions appeared in the groups $H^p(K; \Gamma_{p-1})$. The theory was algebraically awkward; and at the time we did not know whether the differentiable structures obtained are compatible. We have since proved that they are compatible, but the algebraic awkwardness remains.

A metaphysical reason for this awkwardness is that we tackled the problems in the wrong order; the basic problem should be that of imposing a differentiable structure. Using this insight, M. Hirsch outlined in [4] an obstruction theory for Problem I that included II as a special case; the obstructions appeared in the same cohomology groups as before.

Hirsch's theory, however, is destined to remain only in outline form. For simultaneously with Hirsch's work, Milnor was constructing his theory of microbundles, and Hirsch and B. Mazur soon saw how they could utilize this theory to obtain an even better way of approaching the problems. We now turn to a consideration of this approach.

3. THE BUNDLE-THEORETIC APPROACH

It seems hardly necessary to remind the reader of the fundamental importance of Milnor's theory of microbundles [13], [14] for the study of the relationship between topological, PL (piecewise-linear), and differentiable manifolds. Nor need we recall the definition of this concept. It will suffice for descriptive purposes to think of a PL microbundle as simply a fibre bundle whose fibre is R^n and whose group is the group PL_n of all PL homeomorphisms of R^n that preserve the origin.

The relevance of microbundles to differentiable structures is based on three facts: First, any vector bundle η over a complex has an underlying PL microbundle $|\eta|$, which one can think of as having been obtained from η by expanding the group of the bundle from the orthogonal group O_n to PL_n. Second, any PL manifold K has a tangent PL microbundle t_K. And third, any differentiable manifold K_α has a tangent vector bundle T_α whose underlying PL microbundle is t_K.

Putting these facts together, one sees that if K has a differentiable structure α, then the tangent PL microbundle of K must be the underlying microbundle of some vector bundle over K. The crucial connection between microbundle theory and manifolds is afforded by the converse:

THEOREM (Milnor). *If the tangent* PL *microbundle of* K *is the underlying microbundle of some vector bundle* η *over* K, *then* K *has a differentiable structure* α. [In fact, all that is required is that t_K be *stably* equivalent to $|\eta|$ for some η (in the sense of Whitney sum); and the conclusion can be strengthened to require that T_α and η be stably equivalent.]

The proof involves the construction of a differentiable manifold E, depending on η, such that K may be imbedded in E with trivial normal microbundle. The latter specification means that there is a piecewise-differentiable homeomorphism of $K \times R^q$ onto an open subset of E. Hence $K \times R^q$ inherits a differentiable structure from this imbedding. One then applies the following theorem of Hirsch, which he had proved as a tool in constructing his obstruction theory:

PRODUCT THEOREM [1], [3]. *If* $K \times R^q$ *has a differentiable structure, then so does* K.

Milnor's existence theorem reduces Problem I to a microbundle-theoretic question; its application to the concordance problem, however, demands a more careful analysis of the way in which the differentiable structure α depends on the vector bundle η. Hirsch and Mazur have carried out this analysis, and have shown that the concordance class of α depends only on the stable class of η. Using this, they have proved the following [5]:

THEOREM. *There is an* h-*space* Γ *(homotopy commutative and homotopy associative) such that*

(a) *for each* PL *manifold* K, *there exists a bundle* ξ_K *over* K *having fibre* Γ *such that* C(K) *is in one-to-one correspondence with the homotopy classes of cross sections of* ξ_K,

(b) *If* ξ_K *has a cross section, then it is fibre-homotopically trivial, and* C(K) *is in one-to-one correspondence with the set* [K, Γ] *of homotopy classes of maps of* K *into the fibre* Γ.

A plausibility argument for this theorem goes as follows: Let BPL and BO be the classifying spaces for stable PL microbundles and stable vector bundles, respectively. Then ζ: BO $\to$ BPL is a fibre bundle with fibre $\Gamma = \lim PL_n/O_n$. If f: $K \to BPL$ is the classifying map for the stable tangent microbundle of K, then Milnor's theorem says that K has a differentiable structure α if and only if f may be lifted to a map of K into BO; the Hirsch-Mazur result says that the concordance class of α depends only on the homotopy class of this lifting. But a lifting of f is equivalent to a cross-section of the bundle $\xi_K = f^*(\xi)$.

We must emphasize that this argument is a gross oversimplification of the situation. In particular, PL_n is *not* the group we have described, but rather a semi-simplicial analogue of it, which does not even contain the semi-simplicial analogue of O_n. As a result, one must replace PL_n by the semi-simplicial analogue PD_n of the *set* of piecewise-differentiable homeomorphisms of $(R^n, 0)$ (it is not a group); the fibre Γ will in fact be the space $\lim PD_n/O_n$.

The work of Hirsch and Mazur is not yet available; but an independent proof of the one-to-one correspondence stated in (b) of the theorem may be found in [10].

This theorem has some immediate consequences for the concordance problem. One is that if C(K) is not empty, it may be given a group structure (not naturally), since Γ is an h-space. Another is that $\pi_i(\Gamma) = [S^i, \Gamma]$ is in one-to-one correspondence with $C(S^i)$; moreover, it is easy to show that the group operations are the same in the two cases, so that $\pi_i(\Gamma) \simeq C(S^i)$. Finally, it follows directly from homotopy-theoretic arguments (and the fact that $C(S^i) \simeq \Gamma_i$) that

$$C(S^i \times S^j) \leftrightarrow [S^i \times S^j, \Gamma] \simeq \Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j};$$

one uses only the fact that Γ is an h-space with ith homotopy group Γ_i. This computation of $C(S^i \times S^j)$ was announced by Mazur at the Seattle conference.

4. CONNECTIONS BETWEEN THE THEORIES

It is clear that when the Hirsch-Mazur theory is worked out in complete detail and published, it will have a far-reaching effect on the concordance problem, and that it will take the center of the stage as more and more work is done. Let us examine the connections between the two theories we have sketched, to see what role the former theory may still have to play.

The connection is made by means of the corresponding obstruction theories. Given a map $\psi: K \to \Gamma$, one may attempt to construct a homotopy between it and the constant map. If an obstruction to this homotopy appears, it will be in the group $H^p(K; \pi_p(\Gamma)) = H^p(K; \Gamma_p)$, as expected. Homomorphisms Λ^k can be defined by homotopy-theoretic means, and they can be proved to represent the operations Λ^k defined in the other theory. Then one has the relation

$$\text{order } C(K) \leq \text{order } \sum_p H^p(K; \Gamma_p)/\text{images } \Lambda^i,$$

as before. But one can do more. For any complex L, one can by homotopy-theoretic means define homomorphisms

$$\Phi^2: H^p(L; \Gamma_p) \to H^{p+2}(L; \Gamma_{p+1}),$$

$$\Phi^3: H^p(L; \Gamma_p) \cap (\text{kernel } \Phi^2) \to H^{p+3}(L; \Gamma_{p+2})/(\text{image } \Phi^2),$$

and so on, such that the realizable obstructions are precisely those elements of $H^p(K; \Gamma_p)$ that lie in the kernel of Φ^k for all k. Hence the realizable obstructions form a subgroup of H^p, and we can write

$$C(K) \leftrightarrow \sum_p (H^p(K; \Gamma_p) \cap (\text{kernels } \Phi^i))/(\text{images } \Lambda^i).$$

The Λ^i and Φ^i are in fact related; Λ^i is induced from Φ^i *via* the suspension isomorphism. For example, the diagram

$$\begin{array}{ccc} H^{p-2}(K; \Gamma_{p-1}) & \xrightarrow{\Lambda^2} & H^p(K; \Gamma_p) \\ \downarrow & & \downarrow \\ H^{p-1}(SK; \Gamma_{p-1}) & \xrightarrow{\Phi^2} & H^{p+1}(SK; \Gamma_p) \end{array}$$

is commutative.

The problem of computing concordance classes now comes down to the problem of computing the operations Λ^i and Φ^i in the cohomology of K. At this point, the older theory may still be useful. For the Λ^i in the first theory are defined very geometrically, and they have been computed in specific cases, as has the group of realizable obstructions. These computations may at least provide future workers on the concordance problem with some useful information, as they explore the mysteries of the space Γ.

5. A NONTRIVIAL EXAMPLE

We illustrate the point with an additional example of computations carried out within the first theory. It is the next example one might consider after the space $S^i \times S^j$; it has the same cohomology groups as that space, but the operators Λ^i act differently.

THEOREM. *Let* M *be the total space of an* S^j*-bundle over* S^i, *where the characteristic map* α *of the bundle may be pulled back to lie in* $\pi_{i-1}(SO(j-1))$. *Then*

$$C(M) \leftrightarrow \Gamma_i \oplus (\Gamma_j \cap \text{kernel } \tau_\alpha) \oplus (\Gamma_{i+j}/\text{image } \tau_\alpha),$$

where $\tau_\alpha\colon \Gamma_m \to \Gamma_{m+i-1}$ *is a certain twisting homomorphism defined for all* $m \geq j$.

Milnor and Kervaire studied the homomorphism τ_α in their work on the groups Θ_n; they showed it to be nontrivial in a number of cases [7], [9]. A specific case in which something nontrivial occurs in both terms is the nontrivial S^8-bundle over S^2. For according to Milnor, both $\tau_\alpha\colon \Gamma_8 \to \Gamma_9$ and $\tau_\alpha\colon \Gamma_9 \to \Gamma_{10}$ are nonzero homomorphisms, if α is the nonzero element of $\pi_1(SO(7))$.

To define τ_α, one first defines a homomorphism

$$\pi_k(SO(m-1)) \otimes \Gamma_m \xrightarrow{\tau} \Gamma_{m+k};$$

the inclusion $\pi_k(SO(j-1)) \to \pi_k(SO(m-1))$ followed by τ defines τ_α in general. To construct τ, we proceed as follows:

We represent an element of $\pi_k(SO(m-1))$ by a differentiable map $f: S^k \to SO(m-1)$, and we represent an element of Γ_m by a diffeomorphism $g: R^{m-1} \to R^{m-1}$ that is the identity outside a compact set. We then define diffeomorphisms F and G of $S^k \times R^{m-1}$ by the equations

$$F(x, y) = (x, f(x) \cdot y) \quad \text{and} \quad G(x, y) = (x, g(y)).$$

The diffeomorphism $F^{-1}GF$ of $S^k \times R^{m-1}$ is the identity outside a compact set. *Via* the standard imbedding of $S^k \times R^{m-1}$ in S^{m+k-1} that sends (x, y) into $(x, y)/\|(x, y)\|$, it induces a diffeomorphism of S^{m+k-1}.

A final remark: The reader will observe that the definition of τ looks suspiciously similar to G. Whitehead's definition of the J-homomorphism [25]

$$J: \pi_k(SO(m-1)) \to \pi_{m+k}(S^m).$$

The similarity is more than accidental. Using the equivalence of our obstruction theory with that of Hirsch and Mazur, one can see that (up to sign) τ is simply the composite

$$\pi_k(SO(m-1)) \otimes \pi_m(\Gamma) \xrightarrow{J \otimes i} \pi_{m+k}(S^m) \otimes \pi_m(\Gamma) \xrightarrow{\phi} \pi_{m+k}(\Gamma),$$

where ϕ is the composition map carrying $[f] \otimes [g]$ into $[gf]$. Alternatively, one can replace $\pi_m(\Gamma) \simeq \Theta_m$ by its image under the Milnor-Kervaire homomorphism $\Theta_m \to \pi_{m+j}(S^j)$, and prove directly that τ equals the composite of $J \otimes i$ and ϕ in the stable homotopy groups of spheres. The latter method is in fact the one used by Milnor in demonstrating the nontriviality of τ.

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