ε-MAPPINGS AND GENERALIZED MANIFOLDS

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Dedicated to R. L. Wilder on his seventieth birthday.

INTRODUCTION AND RESULTS

The purpose of this paper is to show that n-dimensional absolute neighborhood retracts that admit ε-maps onto closed orientable n-manifolds, for arbitrarily small $\varepsilon > 0$, are necessarily orientable generalized n-manifolds in the sense of Wilder. In a sequel to this paper we shall show that if one omits the orientability hypothesis, then one obtains locally orientable generalized n-manifolds.

All spaces considered are subsets of compact metric spaces. A map $f: X \to Y$ of a space X onto Y is an *ε-map* $(\varepsilon > 0)$ provided $\operatorname{diam} f^{-1}(y) < \varepsilon$, for each $y \in Y$. If Π is a class of compact polyhedra, we say that X is *Π-like* provided for each $\varepsilon > 0$ there exist a polyhedron $P \in \Pi$ and an ε-mapping $f: X \to P$ onto P (P and f depend on ε) (see [14, Definition 1]). By a *(closed)* n-*manifold* we mean a (compact) triangulable manifold without boundary having covering dimension n. We are interested here in Π-like continua, where $\Pi = \mathfrak{M}^n$ is the class of all closed, connected, orientable n-manifolds.

Homology and cohomology modules H_r and H^r are taken in the sense of Čech, based on arbitrary open coverings as in [8]. Given a principal ideal domain L, we say that a compact space X is *homology locally connected up through dimension* n *over* L (written lc_n^L) provided for each $x \in X$ and each open set $U \subset X$ about x there exists an open set V about x $(V \subset U)$ such that

$$i_r^{VU} = 0 \qquad (0 \le r \le n),$$

where

$$i_r^{VU}: H_r(V; L) \to H_r(U; L)$$

is the homomorphism induced by inclusion $i_{VU}: V \to U$. In dimension zero we use augmented homology. It is well known that every locally contractible space X, and *a fortiori* every ANR, is lc_n^L for each L.

For open sets U of a compact space X we consider cohomology modules with compact supports

$$H_c^r(U; L) = H^r(X, X \setminus U; L)$$

(see for example [20, p. 248]). For open sets U and V $(V \subset U)$, the inclusion map $i_{VU}: V \to U$ induces homomorphisms $i_{VU}^r: H_c^r(V; L) \to H_c^r(U; L)$.

The rth local co-Betti number at $x \in X$ over L, denoted by $p^r(x, X; L)$, is defined as follows: $p^r(x, X; L) = k$ means that for each open set U about x there exist

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open sets W and V ($x \in W \subset V \subset U$) such that for every open set W' ($x \in W' \subset W$) the modules

$$i^r_{W'V}(H^r_c(W'; L)) \quad \text{and} \quad i^r_{WV}(H^r_c(W; L))$$

coincide and are free L-modules of rank k (see for example [2, p. 7, Definition 2.1]). In case L is a field F, the space X is of finite cohomology dimension $\dim_F X = n$, and X is lc^F_n, then $p^r(x, X; F)$ is simply the dimension of the limit of the inverse system $\{H^r_c(U; F); i^r_{VU}\}$, where the sets U range over all open neighborhoods of x in X (see [21] and [2, Section 2, pages 7 and 8]).

By an orientable, n-dimensional, generalized closed manifold over L (n-gcm_L) we mean a continuum X with the following properties:

(i) $\dim_L X$ is finite,

(ii) $p^r(x, X; L) = 0 \quad (r \neq n)$,

(iii) $p^n(x, X; L) = 1$,

(iv) $H^n(X; L) = L$ and each $x \in X$ has a basis of connected open neighborhoods U for which $i^n_{UX}(H^n_c(U; L)) = H^n(X; L)$ (see [20, pp. 244, 250] and [2, pp. 9 and 12]).

The following is our main result.

THEOREM 1. *Let* X *be an* n-*dimensional absolute neighborhood retract that is* $\mathfrak{M}^n$-*like. Then* X *is an orientable* n-gcm_L *over every principal ideal domain* L.

This theorem gives a positive answer (in the orientable case) to a problem raised by T. Ganea [11] and also by H. Cartan (private communication from Ganea). In case n = 2, a well-known theorem of R. L. Wilder [20, p. 272, Theorem 2.3] implies that X is actually a 2-manifold. This was previously discovered by Ganea [9]. We point out that in [11] Ganea produced a 3-dimensional absolute neighborhood retract that is like the 3-sphere S^3 but fails to be a manifold.

In the case when X is a polyhedron and L is the ring Z of integers, Theorem 1 yields the following result.

COROLLARY 1. *Let* X *be an* n-*dimensional,* $\mathfrak{M}^n$-*like polyhedron. Then* X *is an orientable,* n-*dimensional* h-*manifold* (as defined in [1, Vol. 3, p. 4]).

In [6], A. Deleanu has shown that an n-dimensional polyhedron that admits ε-maps onto closed n-manifolds, for all $\varepsilon > 0$, is necessarily a closed pseudo-manifold, and that if $n \leq 3$, then it is a manifold. On the other hand, it is known that an n-dimensional h-manifold is a closed pseudo-manifold [1, Vol. 3, p. 5, Theorem 1.22], and that for $n \leq 3$ it is a closed manifold (see [1, Vol. 3, p. 7, 1.3]). Thus, Corollary 1 is a considerable strengthening of Deleanu's results (in the orientable case). P. M. Rice has recently exhibited, for each $n \geq 4$, an n-dimensional polyhedron that is S^n-like but fails to be a manifold [18].

An associated problem is that of quasi-embeddability. We say that a compact space X *quasi-embeds* in Y provided for each $\varepsilon > 0$ there exists an ε-mapping of X into Y. Ganea [10] has shown that if an n-dimensional polyhedron ($n \neq 2$) quasi-embeds in Euclidean space R^{2n}, then it embeds in R^{2n}. We have shown in [15] that for $n \leq 2$, n-dimensional polyhedra embed in R^n if they quasi-embed in R^n.

THEOREM 2. *For each* $n \geq 4$, *there exists an* n-*dimensional polyhedron that quasi-embeds in* R^n *but fails to embed in* R^n.

We do not know whether Theorem 2 is true for n = 3.

PROOFS AND RESULTS

1. In view of recent results of F. Raymond [17, Theorems 1 and 2] it is sufficient to prove Theorem 1 for the case when L is a field F. Moreover, since the covering dimension dim X is n, we see that

$$\dim_F X \leq \dim_Z X = \dim X = n,$$

so that (i) in the definition of an orientable n-gcm_F is satisfied.

2. We next show that X satisfies (iv) and (iii). It is implicit in Ganea's paper [9] that

$$H^n(X; F) \approx F \approx H_n(X; F). \tag{1}$$

In the first place,

$$H^n(X; Z) \neq 0$$

(see [9, (3.1.1) and 2.1]). Moreover, X being an ANR, every ε-map of X onto an n-manifold $M \in \mathfrak{M}^n$ has a left homotopy inverse, provided ε is small enough [7]. Therefore, $H^n(X; Z)$ is a direct summand of $H^n(M; Z) \approx Z$, and thus

$$H^n(X; Z) \approx Z.$$

Applying the universal coefficient formula

$$H^n(X; F) \approx H^n(X; Z) \otimes F$$

(see [9, 2.1]), we obtain (1).

Deleanu has proved [4, Theorem 1] that for each connected open set $U \subset X$ the homomorphism

$$i^n_{UX}: H^n_c(U; F) \to H^n(X; F)$$

is an isomorphism (onto). This and (1) establish (iv). Moreover, if V and U are connected open sets in X $(V \subset U)$, then

$$i^n_{VU}: H^n_c(V; F) \to H^n_c(U; F)$$

is an isomorphism, because i^n_{VX} and i^n_{UX} are isomorphisms and

$$i^n_{VX} = i^n_{UX} i^n_{VU}.$$

Therefore, if we restrict the inverse system $\{H^n_c(U; F); i^n_{VU}\}$ to the cofinal subsystem determined by connected open neighborhoods U of $x \in X$, we have a system each term of which is F and each projection of which is an isomorphism. Hence, the limit of the system is F, and this establishes (iii).

3. In order to show that (ii) holds, it is enough to prove that, for every $r < n$ and for every open neighborhood U of $x \in X$, there exists an open neighborhood V of x $(V \subset U)$, such that the homomorphism

$$i^r_{VU}: H^r_c(V; F) \to H^r_c(U; F)$$

is zero. This proof will be given in several steps.

We first apply Theorem 1 of [14] to obtain an inverse sequence $\{X_i; \pi_{ij}\}$ ($i = 1, 2, \cdots$) of closed orientable n-manifolds $X_i \in \mathfrak{M}^n$ with maps $\pi_{ij}: X_j \to X_i$ onto X_i ($i \leq j$). This sequence is kept fixed for the rest of the proof. As usual, $\pi_i: X \to X_i$ denotes the projection of X onto X_i. For each $\varepsilon > 0$, the projections $\pi_i: X \to X_i$ are ε-maps, for sufficiently large i. Therefore, for large i, the maps π_i have left homotopy inverses, and therefore

$$\pi_i^*: H^n(X_i; F) \to H^n(X; F) \quad \text{and} \quad \pi_{i*}: H_n(X; F) \to H_n(X_i; F)$$

are isomorphisms.

Since

$$\pi_{ij*}\pi_{j*} = \pi_{i*} \quad (i \leq j),$$

we see that the mappings

$$\pi_{ij*}: H_n(X_j; F) \to H_n(X_i; F) \quad (i \leq j),$$

are also isomorphisms, for sufficiently large i. This enables us to select, for each large i, an orientation class

$$\alpha_i \in H_n(X_i; F) \approx F \quad (\alpha_i \neq 0)$$

in such a way that

(2) $$\pi_{ij*}\alpha_j = \alpha_i \quad (i \leq j).$$

4. We denote by $\mathfrak{B}$ the basis for the topology of X consisting of all sets of the form $\pi_i^{-1}(U_i)$, where U_i is open in X_i ($i = 1, 2, \cdots$). Given a (nonempty) set $U \in \mathfrak{B}$ of the form

$$U = \pi_{i_0}^{-1}(U_{i_0}) \quad (U_{i_0} \text{ open in } X_{i_0}),$$

we consider the open sets

$$U_i = \pi_{i_0 i}^{-1}(U_{i_0}) \quad (i_0 \leq i)$$

in X_i. We obtain thus the inverse sequence

$$\{(X_i, X_i \setminus U_i); \pi_{ij}\}$$

of compact pairs $(X_i, X_i \setminus U_i)$, whose limit is the pair $(X, X \setminus U)$.

Open sets $U_i \subset X_i$ are orientable n-manifolds (not necessarily connected).

For each sufficiently large i, we choose an orientation class $\beta_i \neq 0$ from the infinite homology group

$$\mathfrak{H}_n(U_i; F) = H_n(X_i, X_i \setminus U_i; F)$$

(see [20, p. 248]) in such a way that

(3) $$\beta_i = \mu(\alpha_i),$$

where

$$\mu\colon H_n(X_i; F) \to H_n(X_i, X_i \setminus U_i; F)$$

is the homomorphism induced by the inclusion $X_i \subset (X_i, X_i \setminus U_i)$. Furthermore, the commutativity of the diagram

$$\begin{array}{ccc} H_n(X_i; F) & \xleftarrow{\pi_{ij*}} & H_n(X_j; F) \\ \mu \downarrow & & \downarrow \mu \\ H_n(X_i, X_i \setminus U_i; F) & \xleftarrow[\pi_{ij*}]{} & H_n(X_j, X_j \setminus U_j; F), \end{array}$$

together with relations (2) and (3), implies that

(4) $$\beta_i = \pi_{ij*}\beta_j \qquad (i \leq j).$$

In other words, the maps π_{ij*} preserve the orientation classes $\beta_i \in \mathfrak{H}_n(U_i; F)$ of U_i.

5. The cap product on an open manifold such as U_i pairs infinite homology $\mathfrak{H}_n(U_i; F)$ and compact cohomology $H_c^r(U_i; F)$ to yield homology $H_{n-r}(U_i; F)$ (see [20, p. 248, 2.9]). Furthermore, it is known that the cap product with the orientation class

$$\beta_i \in \mathfrak{H}_n(U_i; F) \text{ of } U_i$$

defines the Poincaré duality isomorphism

$$\beta_i \frown\, : H_c^r(U_i; F) \to H_{n-r}(U_i; F),$$

(see [20, p. 260, 5.16] or [12]). Moreover, the cap product is a natural operation; that is, for

$$h \in H_c^r(U_i; F) \qquad \text{and} \qquad \beta_j \in \mathfrak{H}_n(U_j; F)$$

we have the relation

(5) $$\pi_{ij*}(\beta_j \frown \pi_{ij}{}^* h) = (\pi_{ij*}(\beta_j)) \frown h \qquad (i \leq j)$$

(see for example [20, p. 157, Theorem 17.1a]).

By (4), the relation (5) becomes

$$\pi_{ij*}(\beta_j \frown \pi_{ij}{}^* h) = \beta_i \frown h \qquad (i \leq j,\ h \in H_c^r(U_i; F)),$$

which proves that the diagram

(6) $$\begin{array}{ccc} H_c^r(U_i; F) & \xrightarrow{\pi_{ij}{}^*} & H_c^r(U_j; F) \\ \beta_i \frown \downarrow & & \downarrow \beta_j \frown \\ H_{n-r}(U_i, F) & \xleftarrow[\pi_{ij*}]{} & H_{n-r}(U_j; F) \end{array}$$

is commutative and that its vertical arrows are isomorphisms.

6. Let $V \in \mathfrak{B}$ be another open set of X from the basis $\mathfrak{B}$, and let $V \subset U$, $V \neq \emptyset$. Everything said about U applies as well to V with the roles of U_i, U_j, β_i, β_j taken by V_i, V_j, γ_i, γ_j, so that we obtain a diagram like (6) for the latter.

Moreover, the diagram

$$(D_1)\qquad \begin{array}{ccc} H_c^r(V_i\,;\,F) & \xrightarrow{i^r_{V_i U_i}} & H_c^r(U_i\,;\,F) \\ \gamma_i\frown\downarrow & & \downarrow\beta_i\frown \\ H_{n-r}(V_i;\,F) & \xrightarrow[i^r_{V_i U_i}]{} & H_{n-r}(U_i;\,F) \end{array}$$

is commutative; here the vertical arrows are isomorphisms of the Poincaré duality law (see [3, p. 20-04], or [19, p. 138]).

Finally, from the naturality of cohomology and homology we have commutative diagrams

$$(7)\qquad \begin{array}{ccc} H_c^r(U_i\,;\,F) & \xrightarrow{\pi_{ij}^*} & H_c^r(U_j\,;\,F) \\ i^r_{V_i U_i}\uparrow & & \uparrow i^r_{V_j U_j} \\ H_c^r(V_i\,;\,F) & \xrightarrow[\pi_{ij}^*]{} & H_c^r(V_j\,;\,F) \end{array}$$

and

$$(8)\qquad \begin{array}{ccc} H_{n-r}(U_i\,;\,F) & \xleftarrow{\pi_{ij*}} & H_{n-r}(U_j\,;\,F) \\ i^{V_i U_i}_{n-r}\uparrow & & \uparrow i^{V_j U_j}_{n-r} \\ H_{n-r}(V_i;\,F) & \xleftarrow[\pi_{ij*}]{} & H_{n-r}(V_j\,;\,F)\,. \end{array}$$

7. Diagram (7), together with the equation

$$(9)\qquad H_c^r(U;\,F) = \text{Dir lim}\,\{H_c^r(U_i\,;\,F);\,\pi_{ij}{}^*\}$$

(continuity of cohomology with compact supports) yields the commutative diagram

$$(D_2)\qquad \begin{array}{ccc} H_c^r(U_i\,;\,F) & \xrightarrow{\pi_i^*} & H_c^r(U;\,F) \\ i^r_{V_i U_i}\uparrow & & \uparrow i^r_{VU} \\ H_c^r(V_i\,;\,F) & \xrightarrow[\pi_i^*]{} & H_c^r(V;\,F)\,. \end{array}$$

Similarly, the inverse sequence $\{H_{n-r}(U_i\,;\,F);\,\pi_{ij*}\}$ has an inverse limit

$$(10)\qquad \tilde{H}_{n-r}(U;\,F) = \text{Inv lim}\,\{H_{n-r}(U_i\,;\,F);\,\pi_{ij*}\}\,,$$

and the homomorphisms

$$i_{n-r}^{V_i U_i}: H_{n-r}(V_i; F) \to H_{n-r}(U_i; F)$$

define a limit homomorphism

$$\tilde{i}_{n-r}^{VU}: \tilde{H}_{n-r}(V; F) \to \tilde{H}_{n-r}(U; F) \tag{11}$$

(see (8)). We now obtain the commutative diagram

$$(D_2') \qquad \begin{array}{ccc} H_{n-r}(U_i; F) & \xleftarrow{\tilde{\pi}_{i*}} & \tilde{H}_{n-r}(U; F) \\ i_{n-r}^{V_i U_i} \uparrow & & \uparrow \tilde{i}_{n-r}^{VU} \\ H_{n-r}(V_i; F) & \xleftarrow[\tilde{\pi}_{i*}]{} & \tilde{H}_{n-r}(V; F). \end{array}$$

Finally, the isomorphisms

$$\beta_i \frown : H_c^r(U_i; F) \to H_{n-r}(U_i; F)$$

define an isomorphism

$$\beta: H_c^r(U; F) \to \tilde{H}_{n-r}(U; F)$$

as follows. By (9), each $h \in H_c^r(U; F)$ is of the form $h = \pi_i^* h_i$, where

$$h_i \in H_c^r(U_i; F).$$

It is enough to define $\tilde{\pi}_{j*} \beta h$, for $j \geq i$. Such a definition is given by

$$\tilde{\pi}_{j*} \beta h = \beta_j \frown (\pi_{ij}^* h_i).$$

From the diagram (6) we obtain the cummutative diagram

$$(D_3) \qquad \begin{array}{ccc} H_c^r(U_i; F) & \xrightarrow{\pi_i^*} & H_c^r(U; F) \\ \beta_i \frown \downarrow & & \downarrow \beta \\ H_{n-r}(U_i; F) & \xleftarrow[\tilde{\pi}_{i*}]{} & \tilde{H}_{n-r}(U; F). \end{array}$$

Similarly, for $V \in \mathfrak{B}$, we obtain an isomorphism

$$\gamma: H_c^r(V; F) \to \tilde{H}_{n-r}(V; F),$$

and replacing U_i, U, β_i, β by V_i, V, γ_i, γ in (D_3), we obtain a commutative diagram (D_3').

8. LEMMA 1. *Let* $V \subset U \subset X$ *be nonempty open sets from the basis* $\mathfrak{B}$. *Then the diagram*

$$\text{(D)}\qquad \begin{array}{ccc} H_c^r(V;F) & \xrightarrow{i_{VU}^r} & H_c^r(U;F) \\ \gamma\downarrow & & \downarrow\beta \\ \tilde{H}_{n-r}(V;F) & \xrightarrow[\tilde{i}_{n-r}^{VU}]{} & \tilde{H}_{n-r}(U;F) \end{array}$$

is commutative, with the vertical arrows β *and* γ *being isomorphisms (for* $r = n$ *we use nonaugmented homology in* $\tilde{H}_0(U; F)$ *and* $\tilde{H}_0(V; F)$*).*

Proof. We must show that

$$\beta i_{VU}^r h = \tilde{i}_{n-r}^{VU} \gamma h \tag{12}$$

for each $h \in H_c^r(V)$. Since both sides of (12) are in the inverse limit $\tilde{H}_{n-r}(U; F)$ (see (10)), it is enough to show that

$$\tilde{\pi}_{i*} \beta i_{VU}^r h = \tilde{\pi}_{i*} \tilde{i}_{n-r}^{VU} \gamma h \tag{13}$$

for sufficiently large i. On the other hand, $H_c^r(V)$ is the direct limit of $H_c^r(V_i)$ (see (9)) so that there is no loss of generality in assuming that h is of the form

$$h = \pi_i^* h_i .$$

Using diagrams (D_2) and (D_3), we see that the left-hand side of (13) is

$$\tilde{\pi}_{i*} \beta i_{VU}^r \pi_i^* h_i = \tilde{\pi}_{i*} \beta \pi_i^* i_{V_i U_i}^r h_i = \beta_i \frown (i_{V_i U_i}^r h_i) .$$

Using diagrams (D_2') and (D_3'), we see that the right-hand side of (13) is

$$\tilde{\pi}_{i*} \tilde{i}_{n-r}^{VU} \gamma \pi_i^* h_i = i_{n-r}^{V_i U_i} \tilde{\pi}_{i*} \gamma \pi_i^* h_i = i_{n-r}^{V_i U_i} (\gamma_i \frown h_i) .$$

Finally, by the diagram (D_1),

$$\beta_i \frown (i_{V_i U_i}^r h_i) = i_{V_i U_i}^r (\gamma_i \frown h_i) .$$

9. LEMMA 2. *Let* $x \in X$ *and* $U \in \mathfrak{B}$ $(x \in U)$; *then there exists a* $V \in \mathfrak{B}$ $(x \in V \subset U)$ *such that the homomorphism* $\tilde{i}_r^{VU}$ *from* (11) *is zero* $(0 \leq r \leq n)$ *(for* $r = 0$ *we use augmented homology).*

Proof. Choose an open set $U' \in \mathfrak{B}$ such that $x \in U' \subset \mathrm{Cl}\, U' \subset U$ and $\mathrm{Cl}\, U'$ is compact. Since X is lc_n^F, there is an open set V' such that $x \in V' \subset U'$ and

$$i_r^{V'U'} = 0 \qquad (0 \leq r \leq n) . \tag{14}$$

Furthermore, we can find an open set $V \in \mathfrak{B}$ $(x \in V \subset \mathrm{Cl}\, V \subset V')$, with $\mathrm{Cl}\, V$ compact, such that

$$\pi_i(V) \subset \pi_i(\mathrm{Cl}\, V) \subset \pi_i(\mathrm{Cl}\, U') \subset \pi_i(U) . \tag{15}$$

Now consider the commutative diagram of inverse sequences (where the vertical columns are induced by (15))

$$
(16)\qquad
\begin{array}{ccccc}
H_r(\pi_i(U); F) & \leftarrow \cdots \leftarrow & H_r(\pi_j(U); F) & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \\
H_r(\pi_i(\mathrm{Cl}\, U'); F) & \leftarrow \cdots \leftarrow & H_r(\pi_j(\mathrm{Cl}\, U'); F) & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \\
H_r(\pi_i(\mathrm{Cl}\, V); F) & \leftarrow \cdots \leftarrow & H_r(\pi_j(\mathrm{Cl}\, V); F) & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \\
H_r(\pi_i(V); F) & \leftarrow \cdots \leftarrow & H_r(\pi_j(V); F) & \leftarrow & \cdots
\end{array}
$$

The limits of the second and third rows are the Čech homology groups $H_r(\mathrm{Cl}\, U'; F)$ and $H_r(\mathrm{Cl}\, V; F)$, respectively, because of the continuity of Čech theory for compact spaces. The limits of the first and fourth rows are the groups $\tilde{H}_r(U; F)$ and $\tilde{H}_r(V; F)$, respectively, defined in (10).

We thus obtain from (16) homomorphisms

$$(17)\qquad \tilde{H}_r(V; F) \to H_r(\mathrm{Cl}\, V; F) \to H_r(\mathrm{Cl}\, U'; F) \to \tilde{H}_r(U; F).$$

Their composition is the homomorphism $\tilde{i}_r^{VU}$

However,

$$(18)\qquad \mathrm{Cl}\, V \subset V' \subset U' \subset \mathrm{Cl}\, U',$$

so that the homomorphism

$$(19)\qquad H_r(\mathrm{Cl}\, V; F) \to H_r(\mathrm{Cl}\, U'; F)$$

induced by inclusion $\mathrm{Cl}\, V \subset \mathrm{Cl}\, U'$ decomposes into homomorphisms

$$(20)\qquad H_r(\mathrm{Cl}\, V; F) \to H_r(V'; F) \xrightarrow{i_r^{V'U'}} H_r(U'; F) \to H_r(\mathrm{Cl}\, U'; F),$$

all induced by corresponding inclusions (18). Using (14) and (20), we conclude that the homomorphism (19) is zero, which together with (17) proves that the homomorphism $\tilde{i}_r^{VU}$ is zero.

10. We now prove that X satisfies condition (ii) of the definition of an n-gcm_F. Given any $x \in X$ and any open neighborhood U_1 of x, we choose a set $U \in \mathfrak{B}$ $(x \in U \subset U_1)$. By Lemma 2, there is a $V \in \mathfrak{B}$ $(x \in V \subset U)$, such that $i_{n-r}^{VU} = 0$ $(0 \leq r \leq n)$. We may therefore conclude from diagram (D) that $i_{VU}^r = 0$ $(0 \leq r \leq n)$, and therefore

$$i_{VU'}^r = i_{UU'}^r i_{VU}^r = 0 \qquad (0 \leq r < n).$$

This concludes the proof of (ii) and Theorem 1.

11. In order to exhibit the n-dimensional polyhedra P required for the proof of Theorem 2, we use the fact that for every $n \geq 4$ there exists a combinatorial n-manifold M with boundary ∂M having the properties that

M is contractible,

$$\pi_1(\partial M) \neq 1,$$

$$M \times I \approx I^{n+1}$$

(see [16] and [4]).

We define P as the cone $C(\partial M)$ over ∂M. (Note that the sphere-like polyhedra described by Rice and mentioned in the introduction are the suspensions $\Sigma(\partial M)$ [18].)

We now show that P is not embeddable in R^n. Suppose on the contrary that there exists an embedding $\phi: P \to R^n$. Since

$$H_{n-1}(\partial M; Z) = Z,$$

it follows from the Alexander duality theorem that $R^n \setminus \phi(\partial M)$ consists of two components U and V. The open cone $P \setminus \partial M$, being connected, maps into one of these components, say U. Since P is contractible, the complement of $\phi(P)$ is connected (Alexander's duality theorem) and does not meet $\phi(\partial M)$. Therefore, $R^n \setminus \phi(P)$ must be contained in V. Thus,

$$\phi(P \setminus \partial M) = U.$$

This proves that the vertex of the cone $P = C(\partial M)$ has a Euclidean neighborhood, contrary to the fact that $\pi_1(\partial M) \neq 1$.

12. To show that P quasi-embeds in R^n, we consider for each ε $(0 < \varepsilon < 1)$ the decomposition

$$P = P_\varepsilon \cup Q_\varepsilon,$$

where

$$P_\varepsilon = (\partial M \times [1 - \varepsilon, 1])/\partial M \times 1,$$

$$Q_\varepsilon = \partial M \times [0, 1 - \varepsilon].$$

Let

$$h_\varepsilon: Q_\varepsilon \to \partial M \times I$$

be the homeomorphism given by

$$h_\varepsilon(x, t) = (x, t/(1 - \varepsilon)).$$

Let

$$g_\varepsilon: P_\varepsilon \to M \times 1$$

be a map that agrees with h_ε on

$$P_\varepsilon \cap Q_\varepsilon = \partial M \times \{1 - \varepsilon\}.$$

Such a map exists, because M is contractible and P_ε is a cone over $P_\varepsilon \cap Q_\varepsilon$. The maps g_ε and h_ε define a map

$$f_\varepsilon: P \to (\partial M \times I) \cup (M \times 1)$$

with the property that any nondegenerate counterimage (under f_ε) of a point is contained in P_ε.

Finally, $(\partial M \times I) \cup (M \times 1)$ is a proper subset of $\partial(M \times I) = S^n$ and can therefore be considered as a subset of R^n. The maps $f_\varepsilon: P \to R^n$ prove that P is quasi-embeddable in R^n.

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