OPEN MAPS ON HAUSDORFF SPACES

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Dedicated to R. L. Wilder on his seventieth birthday.

In [1], McAuley proved the following theorem.

THEOREM 1. *Suppose that* X *is a compact subset of a metric space* M, $\mathrm{Bd}\,X \neq \emptyset$, $\mathrm{Int}\,X \neq \emptyset$, *and* f *is a light open mapping of* X *into* M *such that*

(1) $f(\mathrm{Int}\,X) = \mathrm{Int}\,f(X)$,

(2) $f(\mathrm{Bd}\,X) = \mathrm{Bd}\,f(X)$,

(3) *the singular set* S_f *has the property that* $f(S_f)$ *does not contain a nonempty set open relative to* f(X),

(4) $f(S_f)$ *does not separate* f(X), *and*

(5) *there exists a nonempty* U *in* X, *open relative to* X, *such that* $f \mid U$ *is one-to-one and* $f^{-1} f(U) = U$.

Then f *is a homeomorphism.*

In this note we show that McAuley's methods yield the same conclusions with weaker hypotheses. Given topological spaces X and Y and a map $f: X \to Y$, we define the sets

$$S_f = \{x \mid f \text{ is not one-to-one on any neighborhood of } x\},$$

$$P = \{y \mid f^{-1}(y) \text{ is nondegenerate}\}.$$

We recall that f is *open* if and only if for each set U open in X the set f(U) is open in f(X). The following three lemmas are known and obvious.

LEMMA 1. S_f *is closed.*

LEMMA 2. *If* X *is a Hausdorff space and* f *is an open map, then* P *is an open subset of* f(X).

LEMMA 3. *If* X *is a compact Hausdorff space,* Y *is a Hausdorff space, and* f *is an open map, then* $P \cup f(S_f)$ *is closed.*

THEOREM A. *Let* $f: X \to Y$ *be an open map of a compact Hausdorff space* X *into a Hausdorff space* Y *such that*

(1) $f(S_f) \not\supset P$ *(unless each is empty),*

(2) $f(S_f)$ *does not separate* f(X), *and*

(3) $P \cup f(S_f) \neq f(X)$.

Then f *is a homeomorphism.*

It is clear that this is a generalization of Theorem 1, since hypotheses (3), (4), and (5) of Theorem 1 imply hypotheses (1), (2), and (3), respectively, of Theorem A.

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Proof. If $P = \emptyset$, then f is one-to-one (and $S_f = \emptyset$). Hence f is a homeomorphism. Assume that $P \neq \emptyset$. Then $f(X) = \overline{P} \cup \overline{\mathscr{C}P}$, where complement and closure are taken in f(X). Also, $f(X) \setminus f(S_f) = [P \setminus f(S_f)] \cup [\overline{\mathscr{C}P} \setminus f(S_f)]$, and this separation into two disjoint nonempty open sets provides a contradiction. Hence, $P = \emptyset$, and f is a homeomorphism.

Likewise, Theorem 4 of [1] may be made to read as follows.

THEOREM B. *Let* $f\colon X \to Y$ *be an open map of a compact Hausdorff space into a Hausdorff space such that*

(1) $f(S_f) \not\supset P$ *(unless each is empty),*

(2) $f^{-1}(q)$ *is degenerate for all* $q \in f(S_f)$, *and*

(3) *if* C *is a component of* $f(X) \setminus f(S_f)$, *there exists* $p \in C$ *such that* $f^{-1}(p)$ *is degenerate.*

Then f *is a homeomorphism.*

The proof, by contradiction, is also a separation argument, applied to any component of $f(X) \setminus f(S_f)$ that contains a point of P.

Condition (1) of Theorems A and B is satisfied, for example, if $f(S_f)$ contains no nonempty open subset of f(X).

Our argument, applied to Theorems 2 and 3 of [1], permits us to omit the corresponding conditions (1) and (2), together with the assumption that f is light.

McAuley asks whether Theorem 1 remains valid if condition (3) is deleted. The following example shows that the answer is negative.

Let M be the real line (with open-interval topology), C the Cantor set on the closed interval [0, 1], and $X = C \cup [2, 3]$. Define $f\colon X \to M$ by

$$f(x) = 1 \text{ for } x \in C \quad \text{and} \quad f(x) = x \text{ for } x \in [2, 3].$$

Then f satisfies all hypotheses of Theorem 1 except (3), but not its conclusion. In the plane, one may also construct an example in which X is a connected set.

REFERENCE

1. L. F. McAuley, *Concerning a conjecture of Whyburn on light open mappings*, Bull. Amer. Math. Soc. 71 (1965), 671-674.

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