MAXIMALLY ALMOST PERIODIC AND
UNIVERSAL EQUicontinuous MINIMAL SETS

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Let $T$ be a topological group. A transformation group $(X, T)$ with compact phase space $X$ is called a \textit{universal equicontinuous minimal set for $T$} if $(X, T)$ is equicontinuous minimal and if each equicontinuous minimal transformation group $(Y, T)$ with compact phase is a homomorphic image of $(X, T)$. A metatheorem concerning universal "admissible" minimal sets \cite[Theorem 2]{2} guarantees the isomorphically unique existence of such a universal object for $T$.

Chu \cite{3} has defined in a different way the notion of a universal almost periodic minimal set for $T$, called here a \textit{maximally almost periodic minimal set for $T$} (to avoid ambiguity caused by the equivalence of equicontinuity and almost periodicity for a transformation group with compact phase space).

We discuss below the relations to each other and to group compactifications of $T$ of these two types of objects. If a transformation group $(X, T)$ is a universal equicontinuous minimal set for $T$, then it is a maximally almost periodic minimal set for $T$; we show that the converse holds if $T$ is compact and $(X, T)$ is effective. We prove the existence of many maximally almost periodic minimal sets for noncompact generative $T$ that are not universal equicontinuous minimal sets for $T$. As general references on the category of minimal transformation groups, see \cite{4} and \cite{5}.

All compact and locally compact spaces considered are assumed to be Hausdorff spaces.

For a topological space $X$, $C^*(X)$ denotes the Banach algebra of bounded continuous real-valued functions on $X$ with the uniform norm, and $\phi^*$ denotes the canonical map of $C^*(Y)$ into $C^*(X)$ induced by a continuous map $\phi$ of $X$ into a space $Y$.

Let $T_d$ denote the group underlying $T$ provided with its discrete topology. Let $(C^*(T), T_d, \lambda)$ be the transformation group of isometric automorphisms of $C^*(T)$ given by

$$s((f, t)\lambda) = (ts)f \quad (t, s \in T; \ f \in C^*(T)).$$

Let $A^*(T)$ be the $T_d$-invariant subalgebra of $C^*(T)$ consisting of those functions in $C^*(T)$ that are (left) almost periodic, that is, whose orbits under $\lambda$ are relatively compact. We denote by $\lambda$ again the restriction of $\lambda$ to $A^*(T) \times T_d$.

We call a couple $(X, \phi)$, where $X$ is a compact space and $\phi$ is a continuous map of $T$ into $X$, an \textit{almost periodic compactification of $T$} if $T\phi$ is dense in $X$ and $C^*(X)\phi^* = A^*(T)$, so that $\phi^*$ is an isometric isomorphism of $C^*(X)$ with $A^*(T)$. A \textit{maximally almost periodic minimal set for $T$} is then an equicontinuous minimal transformation group $(X, T)$ with compact phase space $X$ such that $(X, \phi)$ is an almost periodic compactification of $T$ for some $\phi$. (Chu’s definition of a universal almost minimal set imposes an additional condition that is superfluous, in view of Corollary 2.) Clearly, any two maximally almost periodic minimal sets for

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T have homeomorphic phase spaces, but Corollary 1 and Theorem 3 show they need not be isomorphic as transformation groups.

A couple \((G, \phi)\) is called, as usual, a \textit{group compactification of} \(T\) if \(G\) is a compact group and \(\phi\) is a continuous homomorphism of \(T\) onto a dense subgroup of \(G\), and \((G, \phi)\) is said to be \textit{maximal} if for each group compactification \((K, \psi)\) of \(T\) there exists a continuous homomorphism \(\theta\) (necessarily unique) of \(G\) onto \(K\) with \(\phi \theta = \psi\). Any two maximal group compactifications of \(T\) are isomorphic in the obvious sense, but the following construction seems most appropriate in the present context.

Let \(M(T)\) be the enveloping semigroup \([4]\) of \((A^\ast(T), T_d, \lambda)\), and let \(\mu_T\), or simply \(\mu\), denote the map \(t \to \lambda^t\) of \(T\) into \(M(T)\). It is well known that \((M(T), \mu)\) is both a maximal group compactification of \(T\) and an almost periodic compactification of \(T\) (see \([7]\) for example).

Corresponding to a group compactification \((G, \phi)\) of \(T\) and a closed subgroup \(H\) of \(G\), let \(H \setminus G\) be the right coset space of \(H\) in \(G\), and define \((H \setminus G, T, \bar{\phi})\) to be the transformation group such that

\[(Hg, t) \bar{\phi} = Hg(t\phi) \quad (g \in G, \; t \in T).\]

When \(H\) is the trivial subgroup of \(G\), \((H \setminus G, T, \bar{\phi})\) reduces to the right transformation group \((G, T, \bar{\phi})\) of \(G\) induced by \(T\) under \(\phi\) \([6, 1.58]\). For example, \((M(T), T_d, \bar{\mu})\) is precisely the enveloping transformation group of \((A^\ast(T), T_d, \lambda)\).

**LEMMA 1** (from \([5, p. 57]\), \([6, 4.48]\)). A necessary and sufficient condition for a transformation group \((X, T, \pi)\) with compact phase space to be equicontinuous minimal is that it be isomorphic to \((H \setminus G, T, \bar{\phi})\) for some group compactification \((G, \phi)\) of \(T\) and some closed subgroup \(H\) of \(G\). In case \(T\) is abelian, an equivalent condition is that \((X, T, \pi)\) be isomorphic to \((G, T, \bar{\phi})\) for some group compactification \((G, \phi)\) of \(T\).

**Remark.** Let \((G, \phi)\) be a group compactification of \(T\), and for each \(g \in G\) let \(\tau_g\) be the left translation in \(G\) by \(g\). Then \(g \to \tau_g\) is an algebraic anti-isomorphism of \(G\) onto the group of automorphisms of \((G, T, \bar{\phi})\).

**LEMMA 2.** Let \((G, \phi)\) and \((K, \psi)\) be group compactifications of \(T\), and let \(\theta\) be a continuous homomorphism of \(K\) onto \(G\) with \(\psi \theta = \phi\). Then

1. \(\theta\) is the unique homomorphism of \((K, T, \bar{\psi})\) onto \((G, T, \bar{\phi})\) mapping the identity of \(K\) onto the identity of \(G\),
2. a map \(\eta: K \to G\) is a homomorphism of \((K, T, \bar{\psi})\) onto \((G, T, \bar{\phi})\) if and only if \(\eta = \theta \tau_g\) for some \(g \in G\),
3. if \((K, T, \bar{\psi})\) is isomorphic to \((G, T, \bar{\phi})\), then \(\theta\) is a homeomorphic isomorphism of \(K\) onto \(G\).

**Proof.** (1) Clearly, \(\theta\) is equivariant with respect to the actions \(\bar{\psi}, \bar{\phi}\). Let \(\theta'\) be a homomorphism of \((K, T, \bar{\psi})\) onto \((G, T, \bar{\phi})\), with \(e \psi \theta' = e \phi\), where \(e\) is the identity element of \(T\). Then \((x \cdot \phi) \theta' = (x \theta') (t \phi)\) for all \(x \in K, t \in T\). Taking \(x = e \psi\), we see that \(\theta'\) coincides with \(\theta\) on \(T \psi\), whence \(\theta' = \theta\).

(2) Let \(\eta\) be a homomorphism of \((K, T, \bar{\psi})\) onto \((G, T, \bar{\phi})\). If \(g = (e \psi \eta)^{-1}\), then \(\eta = \theta \tau_g\), by (1).

(3) follows from (2) and the preceding Remark.

**THEOREM 1.** Let \((X, T, \pi)\) be a transformation group with compact phase space. Then the following are equivalent:
(1) \((X, T, \pi)\) is a universal equicontinuous minimal set for \(T\).

(2) \((X, T, \pi)\) is isomorphic to \((M(T), T, \bar{\mu})\).

(3) For each maximal group compactification \((G, \phi)\) of \(T\), \((X, T, \pi)\) is isomorphic to \((G, T, \bar{\phi})\).

Proof. Let \((K, \psi)\) be a maximal group compactification of \(T\). Since any two universal equicontinuous minimal sets for \(T\) are isomorphic, it suffices to show that \((K, T, \bar{\psi})\) is a universal equicontinuous minimal set for \(T\). Let \((G, \phi)\) be any group compactification of \(T\), and let \(H\) be a closed subgroup of \(G\). By Lemma 1, it is enough to show that \((H \setminus G, T, \bar{\phi})\) is a homomorphic image of \((K, T, \bar{\psi})\). Now the projection of \(G\) onto \(H \setminus G\) is a homomorphism of \((G, T, \bar{\phi})\) onto \((H \setminus G, T, \bar{\phi})\), and \((G, T, \bar{\phi})\) is a homomorphic image of \((K, T, \bar{\psi})\), by Lemma 2 and the maximality of \((K, \psi)\).

Since \((M(T), \mu)\) is an almost periodic compactification of \(T\), Lemma 1 implies that \((M(T), T, \bar{\mu})\) is a maximally almost periodic minimal set for \(T\). Hence we obtain the following three results.

**Corollary 1.** Each universal equicontinuous minimal set for \(T\) is a maximally almost periodic minimal set for \(T\).

**Corollary 2.** If \((X, T)\) is a maximally almost periodic minimal set for \(T\), and \((X, T)\) is an equicontinuous minimal transformation group with compact phase space, then \(X\) is a continuous image of \(X\).

**Corollary 3.** Let \((X, \phi)\) be an almost periodic compactification of \(T\). Then there exists a unique action \(\pi\) of \(T_d\) on \(X\) such that

\[(s t) \phi = (s \phi, t) \pi \quad (s, t \in T);\]

moreover, \((X, T, \pi)\) is a universal equicontinuous minimal set for \(T\).

Proof. Let \(F\) be the inverse of the isomorphism of \(C^*(X)\) onto \(A^*(T)\) defined by \(\phi^*\). There is a unique homeomorphism \(\theta\) of \(X\) onto \(M(T)\), with \(\theta^* = \mu^* F\), whence \(\phi \theta = \mu\). Define \(\pi: X \times T \to X\) by

\[(x, t) \pi = (x \theta, t) \bar{\mu} \theta^{-1} \quad (x \in X, t \in T).\]

Then \((X, T, \pi)\) is a transformation group isomorphic under \(\theta\) to \((M(T), T, \bar{\mu})\), and direct computation gives \((*)\). The stated uniqueness of \(\pi\) follows from the fact that if \(s \in T\) and \(\tau\) is a homeomorphism of \(X\) onto \(X\) with \((s t) \phi = s \phi \tau\) for all \(s \in T\), then \(\tau\) and \(\pi^t\) agree on \(T\).

**Remark.** Let \((G, \phi)\) be a group compactification of \(T\). Then \((G, \phi)\) is a maximal group compactification of \(T\) if and only if it is an almost periodic compactification of \(T\) (and in this case \(\phi\) is the action \(\pi\) given by Corollary 3). If \((G, \psi)\) is an almost periodic compactification of \(T\) for some map \(\psi\), there is no guarantee that \((G, \phi)\) is itself an almost periodic compactification of \(T\).

If \(T\) is maximally almost periodic, that is, if \(A^*(T)\) separates points of \(T\), then \(\mu\) is injective, and by the next lemma, \((M(T), T, \bar{\mu})\) is strongly effective.

**Lemma 3.** Let \((G, \phi)\) be a group compactification of \(T\). Then \((G, T, \bar{\phi})\) is strongly effective if and only if \(\bar{\phi}\) is injective.

**Theorem 2.** Suppose \(T\) is compact, and let \((X, T, \pi)\) be a maximally almost periodic minimal set for \(T\). Then a necessary and sufficient condition for \((X, T, \pi)\) to be a universal equicontinuous minimal set for \(T\) is that it be effective.
Proof. The necessity follows from Theorem 1 and Lemma 3. We show the sufficiency. We may suppose \((X, T, \pi) = (H \setminus G, T, \bar{\phi})\), where \(G, \phi, H\) are as in Lemma 1. By the compactness of \(T\), \(\phi\) is surjective. Since \(\bar{\phi}\) is a universally transitive action, it is strongly effective. Therefore \(t \phi \neq H\) for all \(t \in T\) with \(t \neq e\), \(H = \{e\}\), and \(\phi\) is injective. Hence \(\phi\) is an isomorphism of \((T, T, \tau)\) with \((G, T, \bar{\phi})\), where \(\tau\) is the group operation in \(T\). But \(\mu\) is an isomorphism of \((T, T, \tau)\) with \((M(T), T, \bar{\mu})\).

Example. Let \(G\) be a nondegenerate compact group, let \(I\) be an infinite set, and let \(T = G^I\). Choose an injection \(\sigma : I \to I\) that is not surjective. Define \(\phi\) to be the continuous "shift" endomorphism of \(T\) given by

\[
(g_i \mid i \in I) \phi = (g_{i \sigma} \mid i \in I).
\]

Then \(\phi\) is surjective but not injective, and therefore \((T, T, \bar{\phi})\) is a maximally almost periodic minimal set for \(T\) that is not a universal equicontinuous minimal set for \(T\).

For a locally compact abelian group \(G, G'\) denotes the character group of \(G\), and \(\phi' : H' \to G'\) denotes the adjoint of a continuous homomorphism \(\phi\) of \(G\) into a locally compact abelian group \(H\); we sometimes identify \(G\) with its second character group \(G''\) under duality.

Let \(T\) be locally compact and abelian. If \(G\) is the universal Bohr compactification \(((T')')_d\) of \(T\), and \(\phi : T \to G\) is the adjoint of the identity map of \((T')_d\) into \(T'\), then \((G, \phi)\) is both a maximal group compactification and an almost periodic compactification of \(T\) \([1]\). Hence we identify \((G, \phi)\) with \((M(T), \mu)\).

**THEOREM 3.** Let \(T\) be generative (locally compact, compactly generated, abelian) and noncompact. Then there exist uncountably many pairwise nonisomorphic, strongly effective, maximally almost periodic minimal sets for \(T\) that are not universal equicontinuous minimal sets for \(T\).

**Proof.** By a structure theorem of Weil \([8, p. 110]\), \(T\) has the form \(R^m \times Z^n \times K\), where \(R\) is the line group, \(Z\) is the discrete group of integers, \(m\) and \(n\) are nonnegative integers, and \(K\) is compact and abelian. Since \(T\) is not compact, \(T = V \times S\), where \(V = R\) or \(V = Z\) and \(S\) is locally compact and abelian. Identify \(M(T)\) with \(M(V) \times M(S)\), so that \(\mu_T = \mu_V \times \mu_S\).

Let \(H\) be a Hamel basis for \(R_d\), with \(1 \in H\). There exists an uncountable collection \(\mathcal{B}\) of subsets of \(H\) such that

1. \(\mathcal{B}\) is totally ordered under inclusion,
2. each member of \(\mathcal{B}\) is equipotent to \(H\),
3. \(H \in \mathcal{B}\),
4. \(B \in \mathcal{B}\) implies \(1 \in B\).

For each \(B \in \mathcal{B}\), let \(D(B)\) be the \(Q\)-submodule of \(R_d\) generated by \(B\), where \(Q\) is the field of rationals.

Consider first the case \(V = R\). Identify \(R'\) with \(R\), as usual, so that \(\mu_V\) is the adjoint of the identity map \(i : R_d \to R\). For each \(B \in \mathcal{B}\), let

\[
G(B) = D(B)' \times M(S), \quad \phi_B = (j_B 1)' \times \mu_S : T \to G(B),
\]

where \(j_B : D(B) \to R_d\) is the inclusion map. Then \((G(B), \phi_B)\) is a group compactification of \(T\) for each \(B \in \mathcal{B}\). In particular, \((G(H), \phi_H) = (M(T), \mu_T)\).
Let $B \in \mathcal{B}$. By Kronecker's approximation theorem, the image of $D(B)$ under $j_{B,i}$ is dense in $R$, whence $(j_{B,i})'$ is a monomorphism. Since $\mu_S$ is injective, it follows from Lemma 3 that $(G(B), T, \phi_B)$ is strongly effective. By (2), $D(B)$ is isomorphic with $R_d$. Therefore $D(B)'$ is homeomorphically isomorphic with $M(R)$, and $G(B)$ is homeomorphic with $M(T)$. Hence $(G(B), T, \phi_B)$ is a maximally almost periodic minimal set for $T$.

Let $A, B \in \mathcal{B}$, with $A \neq B$. We show that $(G(A), T, \phi_A)$ is not isomorphic with $(G(B), T, \phi_B)$. In view of (1), we may assume $B \subseteq A$. If $k: D(B) \to D(A)$ is the inclusion map, and $i_S$ is the identity map of $M(S)$, then $k' \times i_S$ is a continuous homomorphism of $G(A)$ onto $G(B)$, with $\phi_A(k' \times i_S) = \phi_B$. Since $k$ is not surjective, $k' \times i_S$ is not injective. Now apply Lemma 2.

Taking $A = H$ above, we conclude that $(G(B), T, \phi_B)$ is not isomorphic with $(M(T), T, \mu)$ for any $B \in \mathcal{B}$ distinct from $H$.

Suppose now that $V = Z$. Then $\mu_V$ is the adjoint of the identity map $j: R_d/Z \to R/Z$. For each $B \in \mathcal{B}$, $D(B) \supset Z$ by (4), and we take

$$G(B) = (D(B)/Z)' \times M(S), \quad \phi_B = (k_B j)' \times \mu_S: T \to G(B),$$

where $k_B$ is the inclusion map of $D(B)/Z$ into $R_d/Z$. The arguments for the case $V = R$ may now be repeated.

REFERENCES


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