ε-TAMING IN CODIMENSION 3

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Let h: M → Int N be a homeomorphism of a closed combinatorial k-manifold into the interior of a combinatorial n-manifold. For $n \geq 2k + 2$, Cantrell and Edwards [2] have shown that h is locally flat if h is almost locally flat, and Gluck [5] has shown that h is ε-tame if it is locally flat. More recently, Černavskiĭ [3] announced that if h: M → E^n is almost locally flat, then h is ε-tame if $n > \frac{3}{2}k + 1$, and h is locally flat if $n - k \neq 2$ and $n \geq 5$. Our main result is that h: M → Int N is ε-tame if h is almost locally piecewise linear and $n - k \geq 3$. Furthermore we avoid using Černavskiĭ's result in this codimension. The author would like to thank T. Homma for several enlightening discussions.

Before proceeding with the proof, we give the following definitions. By a *manifold* we mean a combinatorial manifold with boundary. Let Int N and Bd N denote the interior and boundary, respectively, of the manifold N. If A is a closed subset of N and ε is a positive number, then by an *ε-push* of (N, A) we mean an isotopy H: $N \times [0, 1] \to N$ such that H_0 is the identity mapping 1, for $t \in [0, 1]$ the restriction $H_t \mid N - U_\varepsilon(A)$ is the identity, and $\operatorname{diam}[H(x \times [0, 1])] < \varepsilon$ for all $x \in N$. An embedding h: M → Int N is said to be *ε-tame* if for each $\varepsilon > 0$ there exists an ε-push H^ε of (N, h(M)) such that $H_1^\varepsilon h$: M → N is piecewise linear (pwl). The embedding h is said to be locally piecewise linear at $p \in M$ if there exists a neighborhood U of p such that $h \mid U$ is pwl.

Denote by $B^n(r)$ the set

$$\{(x_1, x_2, \cdots, x_n) \in E^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\}.$$

Let A be an n-cell, B a k-cell, with Bd B ⊂ Bd A and Int B ⊂ Int A; then (A, B) is a *cell pair of type* (n, k). The cell pair (A, B) of type (n, k) is said to be *trivial* if (A, B) is homeomorphic to $(B^n(1), B^k(1))$. (Throughout the discussion, we consider E^m as being embedded in the natural way in E^n). A homeomorphism h: M → Int N is said to be *locally flat* at $p \in$ Int M if there is a neighborhood U of p in M and a neighborhood V of h(p) in N such that $(V, V \cap h(M)) = (V, h(U))$ is a trivial cell pair of type (n, k).

For reference we state Zeeman's engulfing theorem.

THEOREM (Zeeman [11]). *Let* M *be a* k-*connected* m-*manifold* ($k \leq m - 3$), *and let* X *and* K *be subpolyhedra of* Int M *such that* X *is collapsible and* K *is* k-*dimensional. Then there exists a* (k + 1)-*dimensional subpolyhedron* L *with* K ⊂ L ⊂ Int M *such that* X ∪ L *is collapsible.*

Denote by S(f) the singular set $\operatorname{Cl}\{x \in M \mid f^{-1}f(x) \neq x\}$ of the map f: M → N, and by C(P) the abstract cone over the polyhedron P.

LEMMA 1. *Let* h: △ → E^n *be a homeomorphism of a* k-*simplex into* E^n *such that* h *is locally* pwl *except at* $p \in$ Int △. *If* $n - k \geq 3$, *then there exists a*

Received May 9, 1966.
This paper is part of the author's doctoral dissertation at Florida State University, prepared under the supervision of O. G. Harrold, while the author was supported by NSF Grant GP 4006.

homeomorphism $g: \triangle \cup C(\mathrm{Bd}\,\triangle) \to E^n$ *such that* g *is locally* pwl *except at* p, *and there exists a neighborhood* U *of* p *such that* $g \mid U = h \mid U$.

Proof. Our technique is that used by Irwin in [6]. Let S denote $\triangle \cup C(\mathrm{Bd}\,\triangle)$, and let $f'\colon S \to E^n$ be a map such that $f' \mid \triangle = h$ and $f' \mid C(\mathrm{Bd}\,\triangle)$ is pwl. By Theorem 18 of [10], there is a map $f\colon S \to E^n$ such that $f(C(\mathrm{Bd}\,\triangle)) \cap \{h(p)\} = \emptyset$, $f \mid \triangle = h \mid \triangle$, and $f \mid S - \{p\}$ is in general position; therefore

$$\dim S(f) = s \le 2k - n \le k - 3 \quad \text{and} \quad p \notin S(f).$$

We shall construct collapsible sets C and C' such that

$$S(f) \subset C \subset S - \{p\}, \qquad C' \subset E^n - \{h(p)\}, \qquad f^{-1}(C' \cap f(S)) = C.$$

It will then be possible to extend $f \mid \mathrm{Bd}\,B\colon \mathrm{Bd}\,B \to \mathrm{Bd}\,B'$ conewise to a pwl homeomorphism $g'\colon B \to B'$, where B and B' are suitably chosen regular neighborhoods of C and C', respectively. The desired homeomorphism g will be defined by the conditions $g \mid S - B = f \mid S - B$ and $g \mid B = g' \mid B$.

We shall prove inductively that for $1 \le i \le s + 1$ there exist collapsible sets $C_i \subset S - \{p\}$ and $C_i' \subset E^n - \{h(p)\}$ such that $S(f) \subset C_i$, $f(C_i) \subset C_i'$, and $D_i = f^{-1}(C_i' \cap f(S))$ has dimension not exceeding $s - i$.

Since $S(f)$ is a compact subset of $S - \{p\}$, there exists a k-ball B_1 such that $S(f) \subset \mathrm{Int}\,B_1 \subset B_1 \subset S - \{p\}$. Applying Zeeman's engulfing theorem, replacing (M, m, K, k, X, L) with $(B_1, k, S(f), s, x \in S(f), C_1)$, we obtain a collapsible set $C_1 \subset \mathrm{Int}\,B_1$ such that $S(f) \subset C_1$ and $\dim C_1 = s + 1$. Let B_1' be an n-ball such that

$$f(C_1) \subset \mathrm{Int}\,B_1' \subset B_1' \subset E^n - \{h(p)\}.$$

Since $s + 1 \le n - 3$, we may apply Zeeman's theorem, replacing (M, m, K, k, X, L) with $(B_1', n, f(C_1), s + 1, x \in f(C_1), L_1')$, obtaining a collapsible polyhedron L_1' such that $f(C_1) \subset L_1' \subset E^n - \{h(p)\}$ and $\dim L_1' = s + 2$. Applying Theorem 15 of [10], we obtain a collapsible polyhedron C_1' such that $f(C_1) \subset C_1' \subset E^n - \{h(p)\}$, $\dim C_1' = s + 2$, and $C_1' - f(C_1)$ is in general position with respect to $f(S)$. Therefore

$$\dim[f^{-1}(C_1' \cap f(S)) - C] = \dim[(C_1' - f(C_1)) \cap f(S)] \le s + 2 + k - n \le s - 1.$$

Suppose collapsible sets C_i and C_i' have been constructed such that $S(f) \subset C_i \subset S - \{p\}$, $f(C_i) \subset C_i' \subset E^n - \{h(p)\}$, and $D_i = f^{-1}(C_i' \cap f(S)) - C$ has dimension not exceeding $s - i$. We let B_{i+1} be a k-ball such that

$$C_i \subset \mathrm{Int}\,B_{i+1} \subset B_{i+1} \subset S - \{p\},$$

and we apply Zeeman's theorem, replacing (M, m, K, k, X, L) with

$$(B_{i+1}, k, \mathrm{Cl}[f^{-1}(C_1' \cap f(S)) - C_i], s - i, C_i, L_{i+1}).$$

We thus obtain a polyhedron $L_{i+1} \subset S - \{p\}$ such that $C_{i+1} = C_i \cup L_{i+1}$ is collapsible, $\dim L_{i+1} \le s - i + 1$, and $f^{-1}(C_i' \cap f(S)) \subset C_{i+1}$. Let B_{i+1}' be an n-ball such that $f(C_{i+1}) \subset \mathrm{Int}\,B_{i+1}' \subset B_{i+1}' \subset E^n - \{p\}$. Since $\dim L_{i+1} \le s - i + 1$, we may apply Zeeman's theorem, replacing (M, m, K, k, X, L) with

$$(B_{i+1}', n, \mathrm{Cl}[f(C_{i+1}) - C_i'], s - i + 1, C_i', L_{i+1}').$$

Therefore there exists a polyhedron L'_{i+1} such that $C'_i \cup L'_{i+1}$ is collapsible, $f(C_{i+1}) \subset C'_i \cup L'_{i+1}$ and $\dim L'_{i+1} \leq s - i + 2$. It follows from Theorem 15 of [10] that there exists a polyhedron L''_{i+1} such that $C'_i \cup L''_{i+1}$ is collapsible, $f(C_{i+1}) \subset C'_i \cup L''_{i+1} \subset E^n - \{h(p)\}$, $\dim L''_{i+1} \leq s - i + 2$, and $L''_{i+1} - (C'_i \cup f(C_{i+1}))$ is in general position with respect to $f(S)$. Let $C'_{i+1} = C'_i \cup L''_{i+1}$. Then

$$\dim [f^{-1}(C'_{i+1} \cap f(S)) - C_{i+1}] = \dim [(L''_{i+1} - (C'_i \cup f(C_{i+1}))) \cap f(S)]$$

$$\leq s - i + 2 + k - n \leq s - i + 1 .$$

Therefore we have constructed collapsible sets $C_{i+1} \subset S - \{p\}$ and $C'_{i+1} \subset E^n - \{h(p)\}$ such that $S(f) \subset C_{i+1}$, $f(C_{i+1}) \subset C'_{i+1}$, and

$$D_{i+1} = f^{-1}(C'_{i+1} \cap f(S)) - C_{i+1}$$

has dimension not exceeding $s - i + 1$. This completes the induction.

Let $C = C_{s+1}$ and $C' = C'_{s+1}$. Since

$$f(C) \subset C' \quad \text{and} \quad \dim [f^{-1}(C' \cap f(S)) - C] \leq s - (s + 1),$$

the required collapsible sets have been constructed; this completes the proof of Lemma 1.

LEMMA 2. *Let* $h: M \to \text{Int } N$ *be an embedding of a* k-*manifold into the interior of an* n-*manifold, and let* $n - k \geq 3$. *If* $p \in \text{Int } M$ *is an isolated point at which* h *fails to be locally* pwl, *then* h *is locally flat at* p.

Proof. This lemma generalizes Lemma 2.2 of [2], and the proof is essentially the same. However, for completeness we include the details. Let B be a combinatorial n-ball in N such that $h(p) \in \text{Int } B$, and let C be a combinatorial k-ball neighborhood of p such that $h | C$ is locally pwl except at p and $h(C) \subset \text{Int } B$. Then apply Lemma 1 to $h | C \to \text{Int } B$ to obtain a homeomorphism $g: C \cup C(\text{Bd } C) \to \text{Int } B$ such that g is locally pwl except at p, and such that g and h agree on some neighborhood of p. Then g is locally flat at p if and only if h is locally flat at p. If $n = 4$ and $k = 1$, then by [1] g is locally flat at p. If $n \geq 5$, it follows from Theorem 1 of [9] that g is locally flat except possibly at p, and by Stalling's unknotting theorem [7], $g(C \cup C(\text{Bd } C))$ is trivial in Int B; therefore g is locally flat at p.

THEOREM 1. *Let* $h: M \to \text{Int } N$ *be an embedding of a* k-*manifold into the interior of an* n-*manifold* $(n - k \geq 3)$. *If* $p \in \text{Int } M$ *is an isolated point at which* h *fails to be locally* pwl, *then for each* $\varepsilon > 0$ *there exists an* ε-*push* H *of* $(N, h(p))$ *such that* $H_1 h$ *is locally* pwl *at* p *and at every point at which* h *is locally* pwl.

Proof. It follows from Lemma 2 that h is locally flat at p. Let $B \subset \text{Int } N$ be an n-cell such that $(B, B \cap h(M))$ is a trivial cell pair of type (n, m) and such that $\text{diam } B < \varepsilon$ and $h | A$ is locally pwl except at p, where $A = h^{-1}(B \cap h(M))$. Let $U_0, U_1, \cdots, U_{k-1}$ be neighborhoods of p in Int A, and $V_0, V_1, \cdots, V_{k-2}$ neighborhoods of h(p) in Int B, such that U_i is a k-ball pwl embedded in M, V_i is an n-ball pwl embedded in N, and

$$U_i \subset h^{-1}(\text{Int } V_i \cap h(M)) \subset h^{-1}(V_i \cap h(M)) \subset \text{Int } U_{i+1} \qquad (i = 0, 1, \cdots, k - 2).$$

Since $h | \text{Bd } U_0: \text{Bd } U_0 \to \text{Int } V_0$ is pwl, there exists a map $g: M \to N$ such that $g | M - U_0 = h | M - U_0$, $g | A$ is pwl in general position, and $g(U_0) \subset \text{Int } V_0$. Therefore $S(g) \subset \text{Int } U_1$. By methods similar to those of the proof of Lemma 1, we can inductively construct collapsible sets $C_i \subset \text{Int } U_i$ and $C_i' \subset \text{Int } V_i$ such that $S(g) \subset C_i$, $g(C_i) \subset C_i'$, and $D_i = g^{-1}(C_i' \cap g(M)) - C_i$ has dimension at most $s - i$, where $\dim S(g) = s \leq k - 3$. Therefore there exist collapsible sets $C = C_{s+1}$ and $C' = C_{s+1}'$ such that $S(g) \subset C \subset \text{Int } A$, $C' \subset \text{Int } B$, and $g^{-1}(C' \cap g(M)) = C$.

Taking regular neighborhoods of C and C', we obtain balls R and R' such that

$$S(g) \subset \text{Int } R \subset R \subset \text{Int } A, \quad g(R) \subset R' \subset \text{Int } B, \quad g^{-1}(R' \cap g(M)) = R.$$

Extend $g \mid \text{Bd } R$: $\text{Bd } R \to \text{Bd } R'$ conewise to a pwl homeomorphism $\bar{g}: R \to R'$. We may now define $f: M \to \text{Int } N$ by setting $f \mid R = \bar{g} \mid R$ and $f \mid M - R = h \mid M - R$. Since $R \subset \text{Int } A$, f and h agree on some neighborhood of Bd A, and since f is locally pwl on Int A, (B, f(A)) is a locally flat cell pair of type (n, k). It follows therefore from [4] that (B, f(A)) is trivial. It is now easy to construct an isotopy $H: B \times [0, 1] \to B$ such that H_0 is the identity 1, for $t \in [0, 1]$, $H_t \mid \text{Bd } B = 1 \mid \text{Bd } B$, and $H_1 h \mid A = f \mid A$. Extend H_t to N by the identity on N - B, and the required ε-push is constructed.

THEOREM 2. *Let* M *and* N *be* k- *and* n-*manifolds, respectively, let* M *be compact, and let* $h: M \to \text{Int } N$ *be an embedding that is locally* pwl *except on* $C \subset \text{Int } M$. *If* $n - k \geq 3$ *and* C *is countable, then* h *is* ε-*tame.*

Proof. Theorem 2 follows from Theorem 1 by transfinite induction as in [2].

Tindell [8] has shown that if h is locally flat except on a countable subset of Bd M and locally pwl on Int M, then h is ε-tame. Thus we may improve Theorem 2 in the following way.

COROLLARY. *Let* M *and* N *be* k- *and* n-*manifolds, respectively, let* M *be compact, and suppose* $n - k \geq 3$. *If* $h: M \to \text{Int } N$ *is an embedding that is locally flat on* Bd M *except for a countable set and locally* pwl *on* Int M *except for a countable set, then* h *is* ε-*tame.*

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