ON THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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1. STATEMENT OF RESULTS

We shall show that the trivial estimates $a_n = o(1/\sqrt{n})$ and $b_n = o(1/\sqrt{n})$ for the coefficients of bounded univalent functions and meromorphic univalent functions, respectively, are not essentially best possible.

THEOREM 1. *Let*

$$g(z) = z + b_0 + \cdots + b_n z^{-n} + \cdots$$

be analytic and univalent in $1 < |z| < \infty$. *Then*

$$(1.1) \qquad \int_0^{2\pi} |g'(\rho e^{i\theta})|\, d\theta \leq A\left(1 - \frac{1}{\rho}\right)^{-\frac{1}{2}+\frac{1}{300}} \qquad (1 < \rho < \infty),$$

$$(1.2) \qquad |b_n| \leq A n^{-\frac{1}{2}-\frac{1}{300}},$$

where A *is an absolute constant.*

The only previously known estimate, $|b_n| \leq n^{-1/2}$, follows immediately from the area theorem. In the opposite direction, the first nontrivial result was due to Clunie [1], who constructed a univalent function for which $|b_n| > n^{0.02-1}$ for infinitely many n. This was recently improved [8] to $|b_n| > n^{0.139-1}$.

Let γ be the smallest number such that

$$|b_n| \leq A(\varepsilon) n^{\gamma+\varepsilon-1}$$

for every $\varepsilon > 0$. The estimates above imply that

$$(1.3) \qquad 0.139 \leq \gamma < 0.497.$$

The true value of γ is unknown.

Remark. We can prove an estimate that is slightly stronger than (1.2): For $5 < \lambda < \infty$,

$$(1.4) \qquad \sum_{k=1}^{n} k^\lambda |b_k|^\lambda \leq A(\lambda) n^{\frac{\lambda}{2}-\frac{1}{2}+\frac{18}{\lambda-1}} \qquad (n = 1, 2, \cdots),$$

where $A(\lambda)$ is independent of g.

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For $1 \leq p < \infty$, let $\mathfrak{S}_p$ denote the family of functions $f(z) = z + \cdots$ that are analytic in $|z| < 1$, satisfy the condition $f'(z) \neq 0$, and assume every value at most p times [7, Section 1.3]. In particular, $\mathfrak{S}_1$ is then the family of normalized univalent functions in $|z| < 1$.

THEOREM 2. *Let*

$$f(z) = z + \cdots + a_n z^n + \cdots$$

be a function in $\mathfrak{S}_p$ $(1 \leq p < \infty)$, *and let*

$$f(z) = O((1 - r)^{-\alpha}) \quad (r \to 1 - 0) \tag{1.5}$$

with $\alpha < 1/2$. *Then there exists* $\eta = \eta(\alpha, p) > 0$ *such that*

$$\int_0^{2\pi} |f'(re^{i\theta})| \, d\theta = O\left((1 - r)^{-\frac{1}{2}+\eta}\right) \quad (r \to 1 - 0) \tag{1.6}$$

and

$$a_n = O\left(n^{-\frac{1}{2}-\eta}\right) \quad (n \to \infty). \tag{1.7}$$

This estimate no longer holds for areally mean p-valent functions. For these functions,

$$a_n = o(n^{-1/2}) \tag{1.8}$$

is the best possible estimate for every $\alpha < 1/2$ [6], [9], [3, p. 49]. Hayman [4, p. 392] has raised the question whether (1.8) is best possible also for bounded univalent functions. As Theorem 2 shows, this is false.

In the opposite direction, Littlewood [5] has given an example of a bounded univalent function for which $|a_n| > n^{\sigma - 1}$ for some positive σ and infinitely many n. It is possible to choose $\sigma = 0.139$ [8].

The paper [4] of Hayman gives a recent survey of the theory of coefficients of univalent and multivalent functions. As Hayman points out [4, p. 401], one of the conditions satisfied by univalent functions but not by all multivalent functions is $f'(z) \neq 0$, and we use this condition in the proof of Theorem 2.

2. PROOF OF THEOREM 1

1. Throughout this section, we write $z = r^{-1} e^{i\theta}$ $(0 < r < 1)$. By $A_1, A_2, \cdots$ we denote absolute constants. Let $0 < \delta < 1/4$ and $r = 1/\rho$. From the Schwarz inequality, we obtain the bound

$$\int_0^{2\pi} |g'(z)|^{1+\delta} \, d\theta \leq \left(\int_0^{2\pi} |g'(z)|^2 \, d\theta\right)^{1/2} \left(\int_0^{2\pi} |g'(z)|^{2\delta} \, d\theta\right)^{1/2}. \tag{2.1}$$

By the area theorem,

(2.2) $$\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta = 1+\sum_{\nu=1}^{\infty}\nu^2|b_\nu|^2 r^{2\nu+2} \le \frac{A_1}{1-r}.$$

2. To estimate the last integral in (2.1), we write

(2.3) $$[g'(z)]^\delta = \sum_{k=0}^{\infty} c_k z^{-k} \qquad (|z|>1),$$

where $c_0 = 1$. Then

(2.4) $$\psi(r) = \frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^{2\delta}\,d\theta = \sum_{k=0}^{\infty}|c_k|^2 r^{2k} \qquad (0<r<1).$$

It follows that $r\psi'(r) = 2\sum_{k=1}^{\infty} k\,|c_k|^2 r^{2k}$ and

(2.5) $$r\psi''(r) \le r\psi''(r)+\psi'(r) = 4\sum_{k=1}^{\infty} k^2|c_k|^2 r^{2k-1}$$

A result of Golusin [2, p. 132] shows that

(2.6) $$\left|z\frac{g''(z)}{g'(z)}\right| \le \frac{8|z|^2-2}{|z|^2-1} \le \frac{3}{|z|-1}+8 \qquad (1<|z|\le 1/r_0),$$

where r_0 is some absolute constant $(0<r_0<1)$. (This inequality, with slightly worse constants, also follows from elementary distortion theorems.) From (2.3) and (2.6), we deduce that

$$\sum_{k=1}^{\infty} k^2|c_k|^2 r^{2k+2} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dz}[g'(z)]^\delta\right|^2 d\theta = \frac{\delta^2}{2\pi}\int_0^{2\pi}\left|\frac{g''(z)}{g'(z)}\right|^2 |g'(z)|^{2\delta}\,d\theta$$

$$\le r^2\delta^2\left(\frac{3r}{1-r}+8\right)^2\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^{2\delta}\,d\theta.$$

Using (2.4) and (2.5), we find that

$$\psi''(r) \le 4\delta^2\left(\frac{3}{1-r}+\frac{8}{r}\right)^2\psi(r).$$

An application of Hölder's inequality to (2.4) together with (2.2) and $\delta \le 1/2$ now gives the estimate

$$\psi''(r) \le \frac{36\delta^2}{(1-r)^2}\psi(r)+\frac{A_2}{(1-r)^{3/2}},$$

for $r_0 \le r < 1$. Therefore, integrating by parts, we see that

$$\psi'(r) \leq \psi'(r_0) + \int_{r_0}^{r} \frac{36\delta^2}{(1-t)^2}\,\psi(t)\,dt + \frac{A_3}{(1-r)^{1/2}}$$

$$\leq \frac{A_4}{(1-r)^{1/2}} + \frac{36\delta^2}{1-r}\,\psi(r) - \int_{r_0}^{r} \frac{36\delta^2}{1-t}\,\psi'(t)\,dt\,.$$

By (2.5), the last term is negative. Since $\psi(r) \geq |c_0|^2 = 1$, it follows that

$$\frac{\psi'(r)}{\psi(r)} \leq \frac{A_4}{(1-r)^{1/2}} + \frac{36\delta^2}{1-r}$$

and consequently

$$\psi(r) \leq A_5(1-r)^{-36\delta^2} \qquad (0 \leq r < 1)\,.$$

Therefore, (2.1), (2.2), and (2.4) imply that

$$\int_0^{2\pi} |g'(z)|^{1+\delta}\,d\theta \leq A_6(1-r)^{-\frac{1}{2}-18\delta^2} \tag{2.7}$$

3. We choose $\delta = \frac{1}{72}$, $\beta = \frac{1}{2} - \frac{1}{300}$. Let

$E_1 = E_1(r) = \{\theta\colon |g'(r^{-1}e^{i\theta})| \leq (1-r)^{-\beta}\}$, $\quad E_2 = \{\theta\colon |g'(r^{-1}e^{i\theta})| > (1-r)^{-\beta}\}$.

Then, by (2.7),

$$\int_0^{2\pi} |g'(z)|\,d\theta = \int_{E_1} |g'(z)|\,d\theta + \int_{E_2} |g'(z)|\,d\theta$$

$$\leq \frac{2\pi}{(1-r)^{\beta}} + (1-r)^{\beta\delta} \int_{E_2} |g'(z)|^{1+\delta}\,d\theta$$

$$\leq 2\pi(1-r)^{-\beta} + A_6(1-r)^{-\frac{1}{2}+\beta\delta-18\delta^2}$$

Since $\frac{1}{2} - \beta\delta + 18\delta^2 = \frac{1}{2} - \frac{1}{4\cdot 72} + \frac{1}{300\cdot 72} < \beta$, we have proved (1.1). Applying Cauchy's formula with $\rho = 1 + 1/n$, we immediately obtain (1.2)

4. Finally, we prove (1.4). Let $5 < \lambda < \infty$. The Hausdorff-Young inequality [10, p. 190] implies that

$$\left(\sum_{k=1}^{n} (k\,|b_k|\,r^k)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |rg'(r^{-1}e^{i\theta})|^{\frac{\lambda}{\lambda-1}}\,d\theta\right)^{1-\frac{1}{\lambda}}$$

Let $\delta = 1/(\lambda - 1)$. Then $0 < \delta < 1/4$. Hence (2.7) shows that, for $1/2 < r < 1$,

$$\sum_{k=1}^{n} (k\,|b_k|\,r^k)^\lambda \leq \left[A_6(1 - r)^{-\frac{1}{2} - 18/(\lambda-1)^2}\right]^{\lambda-1}$$

$$\leq A_7(\lambda)(1 - r)^{-\frac{\lambda}{2}+\frac{1}{2} - 18/(\lambda-1)} .$$

We obtain (1.4) by taking $r = 1 - 1/2n$. From (1.4), we get (1.2) by discarding the first $n - 1$ terms in the left member and choosing $\lambda = 73$.

3. PROOF OF THEOREM 2

1. We use $B_1, B_2, \cdots$ to denote constants depending only on p and α, and $K_1, K_2, \cdots$ to denote constants that possibly depend also on f. We choose a pair of positive numbers λ and κ such that $2 < \lambda < \frac{\lambda}{1 - \kappa} < \frac{1}{\alpha}$. Let $0 < \delta < \frac{1}{4}$. From Schwarz's inequality, we obtain (with $z = re^{i\theta}$, $0 < r < 1$) the bound

$$(3.1)\qquad \begin{aligned} J(r) &= \left(\int_0^{2\pi} |f'(z)|^{1+\delta}\, d\theta\right)^2 \\ &\leq \int_0^{2\pi} \frac{|f'(z)|^2}{(1 + |f(z)|)^\lambda}\, d\theta \int_0^{2\pi} (1 + |f(z)|)^\lambda\, |f'(z)|^{2\delta}\, d\theta . \end{aligned}$$

By the Hölder inequality,

$$(3.2)\qquad \begin{aligned} &\int_0^{2\pi} (1 + |f(z)|)^\lambda\, |f'(z)|^{2\delta}\, d\theta \\ &\leq \left(\int_0^{2\pi} (1 + |f(z)|)^{\lambda/(1-\kappa)}\, d\theta\right)^{1-\kappa} \left(\int_0^{2\pi} |f'(z)|^{2\delta/\kappa}\, d\theta\right)^{\kappa} . \end{aligned}$$

2. The family $\mathfrak{S}_p$ is linear-invariant: that is, for every mapping $\omega(z)$ of the unit disk onto itself,

$$f(z) \in \mathfrak{S}_p \Rightarrow \frac{f(\omega(z)) - f(\omega(0))}{\omega'(0) f'(\omega(0))} \in \mathfrak{S}_p .$$

Also, $\mathfrak{S}_p$ is normal [7, Satz 1.3]. Hence [7, Folgerung 1.1], [7, Lemma 1.2]

$$(3.3)\qquad \left|\frac{f''(z)}{f'(z)}\right| \leq \frac{B_1}{1 - r} \qquad (0 \leq r < 1) .$$

(In the case $p = 1$ of univalent functions, this follows at once from the distortion theorems.)

To estimate the last integral in (3.2), we proceed as in the proof of Theorem 1. Let

$$[f'(z)]^{\delta/\kappa} = \sum_{k=0}^{\infty} c_k z^k \qquad (|z| < 1).$$

Then

$$\phi(r) = \frac{1}{2\pi}\int_0^{2\pi} |f'(z)|^{2\delta/\kappa}\, d\theta = \sum_{k=0}^{\infty} |c_k|^2 r^{2k}.$$

For $1/2 < r < 1$, it follows that,

$$\phi''(r) \le 4\sum_{k=1}^{\infty} k^2 |c_k|^2 r^{2k-2} = \frac{2}{\pi}\int_0^{2\pi} \left|\frac{d}{dz}[f'(z)]^{\delta/\kappa}\right|^2 d\theta$$

$$= \frac{2\delta^2/\kappa^2}{\pi}\int_0^{2\pi} \left|\frac{f''(z)}{f'(z)}\right|^2 |f'(z)|^{2\delta/\kappa}\, d\theta .$$

Hence, by (3.3),

$$\phi''(r) \le \frac{B_2\,\delta^2}{(1-r)^2}\,\phi(r).$$

This implies

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(z)|^{2\delta/\kappa}\, d\theta = \phi(r) \le B_3(1-r)^{-B_2\delta^2} \tag{3.4}$$

3. The inequality

$$\int_0^{2\pi} |f(z)|^{\lambda'}\, d\theta \le K_1 + K_1\int_0^r \left(\max_{|z|=t} |f(z)|^{\lambda'}\right) dt$$

(see [3, Theorem 3.2]) together with (1.5) shows that

$$\int_0^{2\pi} (1+|f(z)|)^{\lambda/(1-\kappa)}\, d\theta \le K_2 + K_2\int_0^r (1-t)^{-\alpha\lambda/(1-\kappa)}\, dt \le K_3,$$

because $\lambda/(1-\kappa) < 1/\alpha$. Therefore it follows from (3.2) and (3.4) that (with $\beta = B_4$)

$$\int_0^{2\pi} (1+|f(z)|)^{\lambda}\, |f'(z)|^{2\delta}\, d\theta \le K_4(1-r)^{-\beta\delta^2}. \tag{3.5}$$

It follows from (3.1) and (3.5) that

$$\int_0^1 r(1-r)^{\beta\delta^2} J(r)\,dr \le K_5 \int_0^1 \int_0^{2\pi} \frac{|f'(z)|^2}{(1+|f(z)|)^\lambda}\, r\,d\theta\,dr \le K_6 < \infty$$

(we obtain the last assertion as in [6, p. 291], using the fact that $\lambda > 2$). Since $J(r)$ increases,

$$J(r) \int_r^1 (1-t)^{\beta\delta^2}\, t\,dt \le K_6, \qquad J(r) \le K_7(1-r)^{-1-\beta\delta^2} \qquad (1/2 < r < 1).$$

Hence, by (3.1),

$$\int_0^{2\pi} |f'(z)|^{1+\delta}\, d\theta \le K_8(1-r)^{-\frac{1}{2}-\frac{\beta}{2}\delta^2}. \tag{3.6}$$

Let

$$E_1 = E_1(r) = \{\theta:\ |f'(re^{i\theta})| \le (1-r)^{-1/3}\}, \quad E_2 = \{\theta:\ |f'(re^{i\theta})| > (1-r)^{-1/3}\}.$$

Then as in the proof of Theorem 1, (3.6) implies that

$$\int_0^{2\pi} |f'(z)|\,d\theta \le \frac{2\pi}{(1-r)^{1/3}} + (1-r)^{\delta/3} \int_0^{2\pi} |f'(z)|^{1+\delta}\, d\theta$$

$$\le 2\pi(1-r)^{-\frac{1}{3}} + K_8(1-r)^{-\frac{1}{2}+\frac{\delta}{3}-\frac{\beta}{2}\delta^2}.$$

Since $-\frac{1}{2}+\frac{\delta}{3}-\frac{\beta}{2}\delta^2 > -\frac{1}{2}$ for sufficiently small $\delta > 0$, we have proved (1.6); the estimate (1.7) follows at once.

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