# SUBDOMINANT SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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## 1. PRELIMINARIES

Let us consider differential equations of the form

$$\frac{d^2 y}{dx^2} - P(x)y = 0, \tag{1.1}$$

where x is a complex independent variable,

$$P(x) = x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_{m-1} x + a_m, \tag{1.2}$$

and $a_1, \cdots, a_m$ are complex parameters. Previously, P. F. Hsieh and Y. Sibuya [2], [3] obtained the following result.

THEOREM 1. *Equation* (1.1) *has a solution*

$$y = \mathcal{Y}_m(x, a_1, \cdots, a_{m-1}, a_m) \tag{1.3}$$

*such that* (i) $\mathcal{Y}_m$ *is an entire function of* $(x, a_1, \cdots, a_m)$; (ii) $\mathcal{Y}_m$ *and* $\mathcal{Y}'_m$ *admit respectively the asymptotic representations*

$$\begin{aligned} \mathcal{Y}_m &\cong x^{r_m}\left(1 + \sum_{N=1}^{\infty} B_{m,N}\, x^{-N/2}\right) \times \\ &\qquad \exp\left(-\frac{2}{m+2} x^{(m+2)/2} + \sum_{N=1}^{m+1} A_{m,N}\, x^{(m+2-N)/2}\right), \\ \mathcal{Y}'_m &\cong x^{\frac{m}{2}+r_m}\left(-1 + \sum_{N=1}^{\infty} C_{m,N}\, x^{-N/2}\right) \times \\ &\qquad \exp\left(-\frac{2}{m+2} x^{(m+2)/2} + \sum_{N=1}^{m+1} A_{m,N}\, x^{(m+2-N)/2}\right) \end{aligned} \tag{1.4}$$

*uniformly on each compact set in* $(a_1, \cdots, a_m)$*-space as* x *tends to infinity in any sector of the form*

$$|\arg x| \leq \frac{3\pi}{m+2} - \delta, \tag{1.5}$$

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*where* ' *denotes* d/dx, $\delta$ *is an arbitrary positive number, the quantities* $r_m$, $A_{m,N}$, $B_{m,N}$, *and* $C_{m,N}$ *are polynomials of* $(a_1, \cdots, a_m)$, *and*

$$x^r = \exp\{r(\log|x| + i \arg x)\}$$

*for any constant* r. *Furthermore, if we put*

$$\{x^{-m}P(x)\}^{1/2} = 1 + \sum_{h=1}^{\infty} b_h x^{-h}, \tag{1.6}$$

*we get*

$$r_m = \begin{cases} -\frac{m}{4} & \text{if } m \text{ is odd,} \\ -\frac{m}{4} - b_{1+m/2} & \text{if } m \text{ is even} \end{cases} \tag{1.7}$$

*and*

$$\sum_{N=1}^{m+1} A_{m,N}\, x^{(m+2-N)/2} = - \sum_{1 \leq h < \frac{m}{2}+1} \frac{2}{m+2-2h} b_h x^{(m+2-2h)/2}. \tag{1.8}$$

The solution $\mathscr{Y}_m$ tends to zero as x tends to infinity in any sector of the form

$$|\arg x| \leq \frac{\pi}{m+2} - \delta, \tag{1.9}$$

where $\delta$ is an arbitrary positive number. Hence $\mathscr{Y}_m$ is subdominant in each sector (1.9). Subdominant solutions are uniquely determined by their asymptotic representations at infinity. Hence $\mathscr{Y}_m$ is the unique solution of equation (1.1) that satisfies conditions (i) and (ii) of Theorem 1.

## 2. THE MAIN THEOREM

For each fixed $(x, a_1, \cdots, a_{m-1})$, the quantities $\mathscr{Y}_m$ and $\mathscr{Y}'_m$ are entire functions of $a_m$. In this paper we shall compute their orders. Before we state our result, we want to study the two cases m = 1 and m = 2. If m = 1, equation (1.1) can be written as

$$\frac{d^2y}{dx^2} - (x+\lambda)y = 0,$$

where $\lambda$ is a complex parameter. Hence we get the solution

$$\mathscr{Y}_1(x, \lambda) = 2\sqrt{\pi}\,\mathrm{Ai}(x+\lambda),$$

where Ai is the Airy function of the first kind. This implies that the orders of $\mathscr{Y}_1$ and $\mathscr{Y}'_1$ as entire functions of $\lambda$ are equal to 3/2 (see [1, 10.4.1 (p. 446), 10.4.7 (p. 446), and 10.4.59 (p. 448)]). If m = 2, we get the solution

$$\mathscr{Y}_2(x, 2a, \lambda) = 2^{-\rho/2} \exp\left(\frac{1}{2}a^2\right) D_\rho(\sqrt{2}(x+a)),$$

where $\rho = -(\lambda - a^2 + 1)/2$ and $D_\rho$ is the parabolic cylinder function of Whittaker. This implies that the orders of $\mathcal{Y}_2$ and $\mathcal{Y}_2'$ as entire functions of $\lambda$ are equal to 1 (see [1, 19.3.5 (p. 687) and 19.8.1 (p. 689)]).

In general, if we put $x = z a_m^{1/m}$, equation (1.1) takes the form

$$\frac{d^2 y}{dz^2} - a_m^{(2+m)/m}\{z^m + a_m^{-1/m} a_1 z^{m-1} + \cdots + 1\} y = 0.$$

From this we may surmise that the orders of $\mathcal{Y}_m$ and $\mathcal{Y}_m'$ as entire functions of $a_m$ are equal to $\frac{1}{2} + \frac{1}{m}$, since for $m = 1$ and $m = 2$ this is true. In this paper, we shall prove it for every $m$. Namely, we shall establish the following theorem.

THEOREM 2. *The orders of* $\mathcal{Y}_m$ *and* $\mathcal{Y}_m'$ *as entire functions of* $a_m$ *are equal to* $\frac{1}{2} + \frac{1}{m}$.

It should be remarked that $\frac{1}{2} + \frac{1}{m}$ is less than 1, if $m \geq 3$, and that the order of a polynomial of $\mathcal{Y}_m$ and $\mathcal{Y}_m'$ as an entire function of $a_m$ is not greater than $\frac{1}{2} + \frac{1}{m}$.

## 3. REMARKS

Let us write equation (1.1) in the form

$$\frac{d^2 y}{dx^2} - \{Q(x) + \lambda\} y = 0, \tag{3.1}$$

where

$$Q(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x \tag{3.2}$$

and

$$\lambda = a_m. \tag{3.3}$$

For each complex number $x_0$, let us put

$$Q(x + x_0) = x^m + u_1 x^{m-1} + \cdots + u_{m-1} x + u_m. \tag{3.4}$$

Then the quantities $u_k$ are polynomials of $x_0, a_1, \cdots, a_{m-1}$. Hence the $u_k$ are independent of $\lambda$. Now

$$y = \mathcal{Y}_m(x + x_0, a_1, \cdots, a_{m-1}, \lambda) \tag{3.5}$$

is a solution of the equation

$$\frac{d^2 y}{dx^2} - \{Q(x + x_0) + \lambda\} y = 0. \tag{3.6}$$

On the other hand,

$$y = \mathcal{Y}_m(x, u_1, \cdots, u_{m-1}, u_m + \lambda) \tag{3.7}$$

is another solution of equation (3.6). From (1.4) we can derive the respective asymptotic representations of the two solutions (3.5) and (3.7) as $x$ tends to infinity in any sector (1.5). Then the uniqueness of $\mathcal{Y}_m$ yields the following result.

THEOREM 3. *The function* $\mathcal{Y}_m$ *satisfies the identity*

$$\mathcal{Y}_m(x + x_0, a_1, \cdots, a_{m-1}, \lambda) = K_m \mathcal{Y}_m(x, u_1, \cdots, u_{m-1}, u_m + \lambda), \tag{3.8}$$

*where*

$$K_m = \begin{cases} 1 & \textit{if } m \textit{ is odd}, \\ \exp\left(-\dfrac{2}{m+2} x_0^{(m+2)/2} + \displaystyle\sum_{N=1}^{m+1} A_{m,N} x_0^{(m+2-N)/2}\right) & \textit{if } m \textit{ is even}. \end{cases}$$

In particular, we have the relations

$$(3.9)\qquad \begin{aligned} \mathcal{Y}_m(x_0, a_1, \cdots, a_{m-1}, \lambda) &= K_m \mathcal{Y}_m(0, u_1, \cdots, u_{m-1}, u_m + \lambda), \\ \mathcal{Y}'_m(x_0, a_1, \cdots, a_{m-1}, \lambda) &= K_m \mathcal{Y}'_m(0, u_1, \cdots, u_{m-1}, u_m + \lambda). \end{aligned}$$

The quantities $K_m$, $u_1, \cdots, u_m$ are independent of $\lambda$. Hence the two functions $\mathcal{Y}_m(x_0, a_1, \cdots, a_{m-1}, \lambda)$ and $\mathcal{Y}_m(0, u_1, \cdots, u_{m-1}, u_m + \lambda)$ have the same order as entire functions of $\lambda$. The same is also true for $\mathcal{Y}'_m$. Therefore, in order to prove Theorem 2, it is sufficient to consider the functions $\mathcal{Y}_m(0, a_1, \cdots, a_{m-1}, \lambda)$ and $\mathcal{Y}'_m(0, a_1, \cdots, a_{m-1}, \lambda)$.

## 4. ESTIMATES FOR $|\arg\lambda| \leq \pi - \rho_0$

In this section, we shall derive some estimates of $\mathcal{Y}_m$ and $\mathcal{Y}'_m$ for $|\arg\lambda| \leq \pi - \rho_0$, where $\rho_0$ is an arbitrary positive number. It is easily seen that there exist two positive numbers $r_0$ and $M_0$ such that

$$|P(x)| \geq r_0 |\lambda| \quad \text{and} \quad |P(x)| \geq r_0 x^m \tag{4.1}$$

for

$$x \geq 0, \qquad |\lambda| \geq M_0, \qquad |\arg\lambda| \leq \pi - \rho_0. \tag{4.2}$$

The two numbers $r_0$ and $M_0$ depend only on $\rho_0, a_1, \cdots, a_{m-1}$.

Now let us put

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad A(x) = \begin{bmatrix} 0 & 1 \\ P(x) & 0 \end{bmatrix}.$$

Then equation (1.1) is equivalent to

$$\frac{du}{dx} = A(x)u. \tag{4.3}$$

Put

$$T(x) = P(x)^{-1/4} \begin{bmatrix} 1 & 1 \\ P(x)^{1/2} & -P(x)^{1/2} \end{bmatrix}$$

and

$$R(x) = \begin{bmatrix} 1 & -g(x) \\ g(x) & 1 \end{bmatrix},$$

where

$$\arg [P(x)]^r = r \arg [P(x)] \qquad \text{for any } r$$

and

$$g(x) = \frac{P'(x)}{8[P(x)]^{3/2}}.$$

Then the transformation

(4.4) $$u = T(x)R(x)v$$

reduces equation (4.3) to

(4.5) $$\frac{dv}{dx} = B(x)v$$

with

(4.6) $$B(x) = P(x)^{1/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{S(x)}{1+[g(x)]^2},$$

where

(4.7) $$S(x) = h(x)g(x)\begin{bmatrix} 1 & -g(x) \\ -g(x) & -1 \end{bmatrix} - g'(x)\begin{bmatrix} g(x) & -1 \\ 1 & g(x) \end{bmatrix}$$

and

(4.8) $$h(x) = \frac{P'(x)}{4P(x)}.$$

Let us denote by $\|S(x)\|$ the sum of the absolute values of the components of the matrix $S(x)$. Then we get the inequality

(4.9) $$\|S(x)\| \leq Kx^{-2-\frac{1}{2}m}$$

for $x \geq 1$, $|\lambda| \geq M_0$, $|\arg \lambda| \leq \pi - \rho_0$, where $K$ is a positive constant that depends only on $\rho_0, a_1, \cdots, a_{m-1}$. On the other hand, for each positive number $x_0$, there exists a positive constant $K(x_0)$ such that

(4.10) $$\|S(x)\| \leq K(x_0)\,|\lambda|^{-3/2}$$

for $0 \leq x \leq x_0$, $|\lambda| \geq M_0$, $|\arg \lambda| \leq \pi - \rho_0$. The constant $K(x_0)$ depends also on $\rho_0, a_1, \cdots, a_{m-1}$. The quantity $g(x)$ satisfies similar inequalities, namely,

(4.11) $$|g(x)| \leq Kx^{-1-\frac{1}{2}m}$$

for $x \geq 1$, $|\lambda| \geq M_0$, $|\arg \lambda| \leq \pi - \rho_0$, and

(4.12) $$|g(x)| \le K(x_0)\,|\lambda|^{-3/2}$$

for $0 \le x \le x_0$, $|\lambda| \ge M_0$, $|\arg\lambda| \le \pi - \rho_0$. Therefore

$$\sup_{0 \le x < +\infty} |g(x)| \quad \text{and} \quad \int_0^{+\infty} \frac{\|S(x)\|\,dx}{1+[g(x)]^2}$$

tend to zero as $\lambda$ tends to infinity in the sector $|\arg\lambda| \le \pi - \rho_0$.

Put

$$v = w \exp\left(-\int_0^x [P(t)]^{1/2} dt\right).$$

Then

$$\frac{dw}{dx} = \left\{\begin{bmatrix} 2[P(x)]^{1/2} & 0 \\ 0 & 0 \end{bmatrix} + \frac{S(x)}{1+[g(x)]^2}\right\} w.$$

Letting

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad \frac{S(x)}{1+[g(x)]^2} = \begin{bmatrix} s_{11}(x) & s_{12}(x) \\ s_{21}(x) & s_{22}(x) \end{bmatrix},$$

we get the equations

(4.13) $$\begin{aligned} \frac{dw_1}{dx} &= 2[P(x)]^{1/2} w_1 + s_{11}(x) w_1 + s_{12}(x) w_2, \\ \frac{dw_2}{dx} &= s_{21}(x) w_1 + s_{22}(w) w_2. \end{aligned}$$

Let us transform equations (4.13) into

(4.14) $$\begin{aligned} w_1(x) &= -\int_x^{+\infty} \{s_{11}(t) w_1(t) + s_{12}(t) w_2(t)\} \exp\left(2\int_t^x [P(\sigma)]^{1/2} d\sigma\right) dt, \\ w_2(x) &= 1 - \int_x^{+\infty} \{s_{21}(t) w_1(t) + s_{22}(t) w_2(t)\}\, dt. \end{aligned}$$

Since

(4.15) $$\Re\,[P(x)]^{1/2} > 0$$

for (4.2) if $M_0$ is sufficiently large, equations (4.14) have a solution of the form

$$w_1(x) = E_1(x), \qquad w_2(x) = 1 + E_2(x),$$

where $E_1$ and $E_2$ tend to zero uniformly for $0 \le x < +\infty$ as $\lambda$ tends to infinity in the sector $|\arg\lambda| \le \pi - \rho_0$, and they also tend to zero uniformly for $|\lambda| \ge M_0$, $|\arg\lambda| \le \pi - \rho_0$ as $x$ tends to $+\infty$.

Thus equation (4.5) has a solution of the form

$$v(x) = \begin{bmatrix} E_1(x) \\ 1 + E_2(x) \end{bmatrix} \exp\left( - \int_0^x [P(t)]^{1/2} dt \right).$$

Substituting this solution of (4.5) into (4.4), we get a solution of equation (1.1) of the form

$$\text{(4.16)} \qquad \begin{aligned} y(x) &= \{1 + F_1(x)\} [P(x)]^{-1/4} \exp\left( - \int_0^x [P(t)]^{1/2} dt \right), \\ y'(x) &= \{-1 + F_2(x)\} [P(x)]^{1/4} \exp\left( - \int_0^x [P(t)]^{1/2} dt \right), \end{aligned}$$

where $F_1$ and $F_2$ are similar to $E_1(x)$ and $E_2(x)$. Since $\Re [P(t)]^{1/2} > 0$, this solution is subdominant along the positive real axis. Hence $y(x)$ is a constant multiple of $\mathscr{Y}_m(x, a_1, \cdots, a_{m-1}, \lambda)$. We shall now determine this constant multiplier.

If m is odd, we get

$$\text{(4.17)} \qquad \begin{aligned} y(x) &= \mathscr{Y}_m(x, a_1, \cdots, a_{m-1}, \lambda) \times \\ &\exp\left[ - \int_0^{+\infty} \left( [P(t)]^{1/2} - t^{m/2} - \sum_{1 \le h < \frac{m}{2}+1} b_h t^{\frac{m}{2}-h} \right) dt \right], \end{aligned}$$

since

$$\begin{aligned} \int_0^x [P(t)]^{1/2} dt &= \frac{2}{m+2} x^{(m+2)/2} - \sum_{N=1}^{m+1} A_{m,N} x^{(m+2-N)/2} \\ &+ \int_0^x \left( [P(t)]^{1/2} - t^{m/2} - \sum_{1 \le h < \frac{m}{2}+1} b_h t^{\frac{m}{2}-h} \right) dt. \end{aligned}$$

Then

$$\text{(4.18)} \qquad \begin{aligned} \mathscr{Y}_m(0, a_1, \cdots, a_{m-1}, \lambda) &= \{1 + C_1(\lambda)\} \lambda^{-1/4} \times \\ &\exp\left[ \int_0^{+\infty} \left( [P(t)]^{1/2} - t^{m/2} - \sum_{1 \le h < \frac{m}{2}+1} b_h t^{\frac{m}{2}-h} \right) dt \right], \\ \mathscr{Y}'_m(0, a_1, \cdots, a_{m-1}, \lambda) &= \{-1 + C_2(\lambda)\} \lambda^{1/4} \times \\ &\exp\left[ \int_0^{+\infty} \left( [P(t)]^{1/2} - t^{m/2} - \sum_{1 \le h < \frac{m}{2}+1} b_h t^{\frac{m}{2}-h} \right) dt \right], \end{aligned}$$

where $C_1$ and $C_2$ tend to zero as $\lambda$ tends to infinity in the sector $|\arg \lambda| \leq \pi - \rho_0$. In the integral

$$(4.19) \qquad \int_0^{+\infty}\left([P(t)]^{1/2} - t^{m/2} - \sum_{1\leq h<\frac{m}{2}+1} b_h t^{\frac{m}{2}-h}\right)dt,$$

the variable t is real and positive. However, if we examine the singular points of the integrand and its behaviour at $t = 0$ and $t = \infty$, we easily see that we can take the path of integration along the line $\arg t = \frac{1}{m} \arg \lambda$. Hence let us put $t = \lambda^{1/m}\tau$, where $0 \leq \tau < +\infty$. Then the integral (4.19) can be written as

$$\lambda^{3/2}\int_0^{+\infty}\left((\tau+1)^{1/2} - \tau^{1/2} - \frac{1}{2}\tau^{-1/2}\right)d\tau = -\frac{2}{3}\lambda^{3/2} \qquad \text{if } m = 1,$$

and as

$$\lambda^{\frac{1}{2}+\frac{1}{m}}\left[\int_0^{+\infty}\left((\tau^m+1)^{1/2} - \tau^{m/2}\right)d\tau + O(\lambda^{-1/m})\right] \qquad \text{if } m \geq 3.$$

In a similar manner, if m is even, we get the expressions

$$(4.20) \qquad \begin{aligned} \mathscr{Y}_m(0, a_1, \cdots, a_{m-1}, \lambda) &= \{1 + C_1(\lambda)\}\lambda^{-1/4}\exp L_m(\lambda), \\ \mathscr{Y}'_m(0, a_1, \cdots, a_{m-1}, \lambda) &= \{-1 + C_2(\lambda)\}\lambda^{1/4}\exp L_m(\lambda), \end{aligned}$$

where

$$(4.21) \qquad L_m(\lambda) = \int_0^{+\infty}\left([P(t)]^{1/2} - t^{m/2} - \sum_{h=1}^{m/2} b_h t^{\frac{m}{2}-h} - \frac{b_{1+m/2}}{t+1}\right)dt.$$

In computing $L_m(\lambda)$, we can change the path of integration in the same way as in the case where m is odd. If $m = 2$, then

$$b_{1+m/2} = \frac{1}{2}\left(\lambda - \frac{1}{4}a_1^2\right).$$

Hence

$$L_2(\lambda) = \lambda\left\{-\frac{1}{4}\log\lambda + O(1)\right\}.$$

This implies that $\mathscr{Y}_2$ may be an entire function of $\lambda$ of order 1 and of maximal type. This is evident, since the initial values of the parabolic cylinder functions involve Gamma functions. If $m \geq 4$, the quantity $b_{1+m/2}$ is independent of $\lambda$. Hence

$$L_m(\lambda) = \lambda^{\frac{1}{2}+\frac{1}{m}}\left[\int_0^{+\infty}\left((\tau^m+1)^{1/2} - \tau^{m/2}\right)d\tau + O(\lambda^{-1/m})\right].$$

## 5. ESTIMATES FOR $\pi - \rho_0 \leq \arg\lambda \leq \pi$

In this section, we shall derive some estimates of $\mathscr{Y}_m$ and $\mathscr{Y}'_m$ for $\pi - \rho_0 \leq \arg\lambda \leq \pi$. Similar estimates can be derived also for $-\pi \leq \arg\lambda \leq -\pi + \rho_0$. First of all, put

(5.1) $$\omega = \frac{\pi}{2(m+2)}$$

and assume that

(5.2) $$\rho_0 < 2\omega .$$

Since the direction $\arg x = \omega$ lies in a sector (1.9), the solution $\mathscr{Y}_m$ is subdominant as $x$ tends to infinity in this direction. It is easily seen that there are two positive constants $r_0$ and $M_0$ such that

(5.3) $$|P(x)| \geq r_0 |\lambda| \quad \text{and} \quad |P(x)| \geq r_0 \xi^m$$

for

(5.4) $$x = \xi e^{i\omega}, \quad \xi \geq 0, \quad |\lambda| \geq M_0, \quad \pi - \rho_0 \leq \arg\lambda \leq \pi .$$

The quantities $r_0$ and $M_0$ depend only on $a_1, \cdots, a_{m-1}$.

Let us reduce equation (4.3) to equation (4.5) by the transformation (4.4). Then we get the inequalities

$$\|S(x)\| \leq K\xi^{-2-\frac{m}{2}} \quad \text{and} \quad |g(x)| \leq K\xi^{-1-\frac{m}{2}}$$

for $x = \xi e^{i\omega}$, $\xi \geq 1$, $|\lambda| \geq M_0$, $\pi - \rho_0 \leq \arg\lambda \leq \pi$, where $K$ is a positive constant depending only on $a_1, \cdots, a_{m-1}$. On the other hand, for each positive number $\xi_0$, there is a positive constant $K(\xi_0)$ such that

$$\|S(x)\| \leq K(\xi_0)|\lambda|^{-3/2} \quad \text{and} \quad |g(x)| \leq K(\xi_0)|\lambda|^{-3/2}$$

for $x = \xi e^{i\omega}$, $0 \leq \xi \leq \xi_0$, $|\lambda| \geq M_0$, $\pi - \rho_0 \leq \arg\lambda \leq \pi$. The constant $K(\xi_0)$ depends also on $a_1, \cdots, a_{m-1}$.

Now let us reduce equation (4.5) to (4.13), and then let us consider equations (4.14). The path of integration in the present case is

(5.5) $$x = \xi e^{i\omega} \quad (0 \leq \xi < +\infty) .$$

We want to find an interval of values $\xi$ where

(5.6) $$\Re([P(x)]^{1/2} e^{i\omega}) > 0 .$$

To do this, put

$$P(x) = x^m\{1 + Q(x)\} + \lambda, \quad \dot{\Omega} = \arg\{1 + Q(x)\}, \quad \theta = \arg\lambda .$$

Let $R_0$ be a positive number such that

$$|\arg\{1 + Q(x)\}| \leq \pi/4 \quad \text{and} \quad |1 + Q(x)| \geq 1/2 \quad \text{for } |x| \geq R_0 .$$

Then, for $|x| \geq R_0$, we find that

$$\frac{1}{4}\pi \leq m\omega + 2\omega + \Omega \leq \frac{3}{4}\pi .$$

On the other hand,

$$\pi < \pi + 2\omega - \rho_0 \leq \theta + 2\omega \leq \pi + 2\omega < 2\pi .$$

Let

$$\eta(\lambda) = \left(\frac{-3\,|\lambda|\sin(\theta+2\omega)}{\sin \pi/4}\right)^{1/m}. \tag{5.7}$$

Then

$$\Im\,[P(x)e^{i2\omega}] = \xi^m\,|1+Q(x)|\sin(m\omega+\Omega+2\omega) + |\lambda|\sin(\theta+2\omega) > 0$$

for

$$x = \xi\, e^{i\omega}, \qquad \xi \geq \eta(\lambda), \qquad |\lambda| \geq M_0, \qquad \pi - \rho_0 \leq \arg\lambda \leq \pi \tag{5.8}$$

if $M_0$ is sufficiently large. Therefore the inequality (5.6) holds for (5.8). Therefore equations (4.13) have a solution

$$w_1 = E_1(x), \qquad w_2 = E_2(x) \tag{5.9}$$

such that

$$\lim_{\xi \to +\infty} E_1(\xi\, e^{i\omega}) = 0, \qquad \lim_{\xi \to +\infty} E_2(\xi\, e^{i\omega}) = 1, \tag{5.10}$$

and

$$E_1\,[\eta(\lambda)e^{i\omega}] = O(|\lambda|^{-\sigma}), \qquad E_2\,[\eta(\lambda)e^{i\omega}] = 1 + O(|\lambda|^{-\sigma}), \tag{5.11}$$

where $\sigma$ is a positive constant independent of $\lambda$.

To estimate the quantities $E_1(0)$ and $E_2(0)$, let us put

$$x = |\lambda|^{1/m}\,\tau\, e^{i\omega} \qquad (0 \leq \tau < +\infty). \tag{5.12}$$

Then equations (4.13) take the form

$$\frac{dw_1}{d\tau} = |\lambda|^{1/m} e^{i\omega}\,\{2\,[P(x)]^{1/2} w_1 + s_{11}(x)\,w_1 + s_{12}(x)\,w_2\},$$

$$\frac{dw_2}{d\tau} = |\lambda|^{1/m}\, e^{i\omega}\,\{s_{21}(x)\,w_1 + s_{22}(x)\,w_2\}.$$

On the other hand, x varies from $\eta(\lambda)e^{i\omega}$ to zero as $\tau$ goes to zero from

$$\tau = \left(\frac{-3\sin(\theta+2\omega)}{\sin \pi/4}\right)^{1/m}.$$

Hence it is easily seen that

$$|E_j(0)| \leq k\,\exp\left(h\,|\lambda|^{\frac{1}{2}+\frac{1}{m}}\right) \qquad (j = 1,\ 2) \tag{5.13}$$

for $|\lambda| \geq M_0$, $\pi - \rho_0 \leq \arg \lambda \leq \pi$, where k and h are positive constants.

By inserting (5.9) into (4.4), we get a solution of equation (4.3):

$$u = T(x)R(x)\begin{bmatrix} E_1(x) \\ E_2(x) \end{bmatrix} \exp\left( - \int_0^x [P(t)]^{1/2} dt \right). \tag{5.14}$$

In particular,

$$u(0) = \lambda^{-1/4}\begin{bmatrix} 1 & 1 \\ \lambda^{1/2} & -\lambda^{1/2} \end{bmatrix}\{1_2 + O(|\lambda|^{-3/2})\}\begin{bmatrix} E_1(0) \\ E_2(0) \end{bmatrix},$$

where $1_2$ is the two-by-two unit-matrix. Hence, by the use of (5.13) and the same method as in Section 4, we get estimates of $\mathscr{Y}_m$ and $\mathscr{Y}'_m$ that are similar to the estimates in Section 4. Thus we can complete the proof of Theorem 2.

## REFERENCES

1. M. Abramowitz and I. A. Stegun (Editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards, Applied Mathematics Series 55, Third Printing, March 1965, with corrections; U.S. Government Printing Office, Washington, D.C.

2. P. F. Hsieh and Y. Sibuya, *On the two point connection problem for second order linear ordinary differential equations with polynomial coefficients*, MRC Tech. Sum. Rep. No. 505, Univ. of Wisconsin, September, 1964.

3. ———, *On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients*, J. Math. Anal. Appl. 16 (1966), 84-103.

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