THE JACOBSON RADICAL OF A GROUP ALGEBRA

Michael I. Rosen

Introduction. Let K be a field of characteristic zero, and G a group. It has been conjectured that the group algebra K[G] is semisimple, in other words, that the Jacobson radical of K[G] is zero. If K has elements that are transcendental over the field Q of rational numbers, then K[G] is indeed semisimple (see Amitsur [1]). For the case where K is algebraic over Q, only partial results are known. For example, if G is abelian or G/C is locally finite (C being the center of G), then the conjecture is true (see [1], [4], [5]). We shall give new proofs of these results, and we shall verify the conjecture for a much larger class of groups.

By a *linear representation* p of a group G we shall mean a homomorphism from G to a finite-dimensional linear group over some field. G is said to be *residually linear* if for every $g \in G$ $(g \neq e)$ there exists a linear representation p such that p(g) is not the identity. Our main result is the following.

THEOREM A. *If every finitely generated subgroup of* G *is residually linear, then* K[G] *is semisimple.*

In particular, the result applies to any linear group. Clearly, the property of being residually linear is inherited by subgroups. By the Peter-Weyl theorem, a compact group is residually linear. This gives the following result.

COROLLARY. *Let* H *be a subgroup of a compact group. Then* K[H] *is semisimple.*

The limitations of our methods will be shown in Section 3, where we prove the following proposition.

THEOREM B. *Let* G *be an infinite, finitely generated, simple group. Then* G *has no nontrivial linear representations over any field.*

Graham Higman [2] has shown that there exist groups satisfying the hypotheses of Theorem B.

1. LEMMA 1.1. *Let* S *be a ring, and* $\{I_i\}$ *a collection of two-sided ideals. Suppose* S/I_i *is semisimple for all* i, *and that* $\bigcap_i I_i = (0)$. *Then* S *is semisimple.*

Proof. If S/I_i is semisimple, I_i is the intersection of the maximal left ideals that contain it. Thus $\bigcap_i I_i = (0)$ implies that the intersection of a certain collection of maximal left ideals is (0). This proves the lemma.

LEMMA 1.2. *Let* $\Omega = \{N_i\}$ *be a collection of normal subgroups of* G. *Suppose that for every finite subset* F *of* G *that does not contain* e, *there exists an* $N \in \Omega$ *that does not meet* F. *Let* R *be a ring, and suppose* R[G/N] *is semisimple for all* $N \in \Omega$. *Then* R[G] *is semisimple.*

Received December 23, 1965.

This paper was written while the author held an O.N.R. Research Associateship, ONR 432. The author thanks Professor Earl Lazerson for suggesting the problem under discussion.

Proof. For $N \in \Omega$ consider the natural ring homomorphism π from $R[G]$ to $R[G/N]$. Let $I(N)$ be the kernel of this homomorphism. By Lemma 1.1, we need only prove that $\bigcap I(N) = (0)$, the intersection being over all $N \in \Omega$.

Let $\sum_{j=1}^{s} a(g_j)g_j \in \bigcap I(N)$. Consider the set $F = \{g_j g_k^{-1}\}$ $(1 \leq j, k \leq s$, and $j \neq k)$. By hypothesis there exists an $N \in \Omega$ such that no element of F is in N. If π is the homomorphism associated with N, then $\pi(g_j) \neq \pi(g_k)$ for $j \neq k$. We apply π to $\sum_{j=1}^{s} a(g_j)g_j$ and get the relation $0 = \sum_{j=1}^{s} a(g_j)\pi(g_j)$. Therefore

$$a(g_1) = \cdots = a(g_s) = 0.$$

LEMMA 1.3. *Let* $\Lambda = \{H\}$ *be the set of finitely generated subgroups of* G. *Suppose* $R[H]$ *is semisimple for all* $H \in \Lambda$. *Then* $R[G]$ *is semisimple.*

Proof. Let H be a subgroup of G, and L a maximal left ideal of $R[H]$. We claim there is a maximal left ideal L^* of $R[G]$ such that $L^* \cap R[H] = L$. This follows immediately from the fact that $R[G]$ is a free right $R[H]$ module.

Now suppose $\sum_{i=1}^{s} a(g_i)g_i$ is in the radical of $R[G]$. Let H be the group generated by $g_1, \cdots, g_s$. Then $\alpha = \sum_{i=1}^{s} a(g_i)g_i \in R[H]$. Let L be a maximal left ideal of $R[H]$, and L^* a maximal left ideal of $R[G]$ such that $L^* \cap R[H] = L$. Then $\alpha \in L^*$, and consequently $\alpha \in L$. It follows that α is in the radical of $R[H]$, and thus $\alpha = 0$.

Using the three lemmas, we now prove that $K[G]$ is semisimple for three types of groups. Propositions 1.1 and 1.2 are not new (see [5]).

PROPOSITION 1.1. *If* G *is abelian, then* $K[G]$ *is semisimple.*

Proof. By Lemma 1.3, it suffices to prove the proposition for finitely generated abelian groups.

If G is abelian and finitely generated, G is the direct product of a finite group and a finitely generated free abelian group. Let Ω be the set of subgroups of G of finite index. Ω clearly satisfies the hypotheses of Lemma 1.2 for $R = K$. Thus $K[G]$ is semisimple.

PROPOSITION 1.2. *Let* C *be a central subgroup of* G, *and suppose* G/C *is locally finite. Then* $K[G]$ *is semisimple.*

Proof. Let H be a finitely generated subgroup of G. Then $H/(H \cap C)$ maps monomorphically into G/C and is finite. By Lemma 1.3, we have reduced the problem to the case where G is finitely generated and G/C is finite. Under these assumptions, we first prove that C is finitely generated.

Let $g_1, \cdots, g_s$ be coset representatives for G/C, and let $h_1, \cdots, h_{2t}$ be generators of G such that $h_i^{-1} = h_{i+t}$ for $i = 1, \cdots, t$. Then

$$h_i = g_{\psi(i)}c_i \quad \text{and} \quad g_i g_j = g_{\phi(i,j)}c_{ij},$$

where $c_i, c_{ij} \in C$. Each element in G is a word in the h_i and is thus equal to a coset representative times a product of the c_i and c_{ij}. It now follows easily that the c_i and c_{ij} generate C.

Let Ω be the set of subgroups of finite index in C. Then Ω is simultaneously a set of normal subgroups of finite index in G, and therefore Ω satisfies the hypotheses of Lemma 1.2, for $R = K$. Thus $K[G]$ is semisimple.

PROPOSITION 1.4. *If* G *is a free group, then* K[G] *is semisimple.*

Proof. Let Ω be the set of normal subgroups of G of finite index. It is known that $\bigcap N = (e)$, where the intersection is over all $N \in \Omega$. Moreover, the set Ω is closed under finite intersection. We may once more invoke Lemma 1.2, and the result follows.

2. PROPOSITION 2.1. *Let* L *be a field, and let* $G = GL_n(L)$. *If* $H \subseteq G$, *then* K[H] *is semisimple.*

Proof. By Lemma 1.3, we may confine our attention to finitely generated subgroups of G. Let H be such a subgroup, and suppose $T_1, \cdots, T_m$ generate H. Let A be the algebra generated over the prime field by the coefficients of the matrices $T_1, \cdots, T_m, T_1^{-1}, \cdots, T_m^{-1}$. Then $H \subset GL_n(A)$.

Consider the group $GL_n(A)$. Let P be a maximal ideal of A. We then have a natural homomorphism from $GL_n(A)$ to $GL_n(A/P)$. Let U(P) be the kernel, and consider the set $\Omega = \{U(P)\}$, P ranging over all maximal ideals P of A. We claim that Ω satisfies the first hypothesis of Lemma 1.2. To show this, we note that $U(P) \subseteq I_n + PM_n(A)$, where I_n is the identity matrix and $M_n(A)$ is the algebra of $n \times n$ matrices with coefficients in A. For $i = 1, \cdots, t$, let S_i be an element of $GL_n(A)$, different from I_n. We must show that there exists a $U(P) \in \Omega$ that contains no S_i or equivalently, such that no $S_i - I_n$ belongs to $PM_n(A)$. Let a_i be a nonzero coefficient of $S_i - I_n$, and set $a = a_1 a_2 \cdots a_t$. Since $a \neq 0$ and A is a Noetherian domain, finitely generated over the prime field, the Hilbert Nullstellensatz implies there exists a maximal ideal P such that $a \notin P$. The kernel U(P) satisfies our requirements.

Let $N(P) = U(P) \cap H$. The collection $\Omega' = \{N(P)\}$ satisfies the first hypothesis of Lemma 1.2. Moreover, H/N(P) is isomorphic to a finitely generated subgroup of $GL_n(A/P)$. The problem is reduced to showing that if J is a finitely generated subgroup of $GL_n(A/P)$, then K[J] is semisimple.

By the Hilbert Nullstellensatz, A/P is a finite-dimensional extension of the prime field. If the characteristic of L is not 0, then the prime field is finite, and so is A/P. In this case, $GL_n(A/P)$ is finite, and the result follows immediately. If the characteristic of L is zero, then A/P is an algebraic number field.

Let J be a finitely generated subgroup of $GL_n(L)$, where L is an algebraic number field. Suppose J is generated by $T_1, \cdots, T_m$, and consider the coefficients of $T_1, \cdots, T_m, T_1^{-1}, \cdots, T_m^{-1}$. Since these coefficients are finite in number, there exists a discrete valuation ring 0 of L such that all the coefficients are in 0. Consequently, $J \subset GL_n(0)$. Let P be the maximal ideal of 0. Then $0/P^i$ is finite for all positive integers i. Let U_i be the kernel of the natural homomorphism from $GL_n(0)$ to $GL_n(0/P^i)$. The set $\Omega'' = \{U_i\}$ satisfies the hypotheses of Lemma 1.2, for R = K, and thus K[J] is semisimple, as was to be proved.

PROPOSITION 2.2. *Let* $L_1, L_2, \cdots, L_m$ *be fields, and let*

$$G = GL_{n_1}(L_1) \times \cdots \times GL_{n_m}(L_m).$$

If $H \subseteq G$, *then* K[H] *is semisimple.*

Proof. The arguments given above for the case m = 1 can easily be extended to cover the cases where m > 1. We omit the details.

We are now in a position to prove Theorem A. By Lemma 1.3, we may assume that G is finitely generated. Let $\{p_i\}$ be the set of linear representations of G, and N_i the kernel of p_i. Let Ω be the set of finite intersections of the N_i. If $N \in \Omega$, then G/N is a group to which Proposition 2.2 applies, and thus $K[G/N]$ is semisimple. Suppose $g_i \in G$ and $g_i \neq e$ for $i = 1, \cdots, s$. By hypothesis, there exists an N_i such that $g_i \notin N_i$. It follows that no g_i is in

$$N = N_1 \cap N_2 \cap \cdots \cap N_s \in \Omega .$$

Lemma 1.2 now gives the result.

3. To prove Theorem B, suppose that G satisfies the hypotheses of Theorem B and that p is a nontrivial linear representation. The kernel of p is a normal subgroup of G, not equal to G. Since G is simple, this implies that ker p = (e), and thus that p is faithful. We may thus suppose from the beginning that G is a subgroup of a linear group $GL_n(L)$. Using the notation of Proposition 2.1, we see that there exists an integral domain A in L, finitely generated over the prime field, such that $G \subset GL_n(A)$. Since $\bigcap U(P) = (I_n)$, there exists a maximal ideal P of A such that $G \cap U(P) = (I_n)$. Consequently, G maps monomorphically into $GL_n(A/P)$. In the case where the characteristic of L is not zero, this is a contradiction, since $GL_n(A/P)$ is finite. If the characteristic is zero, A/P is an algebraic number field.

We may thus suppose $G \subset GL_n(L)$, where L is an algebraic number field. Again, from the proof of Proposition 2.1 we see that there exists a discrete valuation ring 0 in L such that $G \subset GL_n(0)$. Since $\bigcap U_i = (I_n)$, there exists an index i such that $G \cap U_i = (I_n)$. Thus G maps monomorphically into $GL_n(0/P^i)$. The latter group is finite, so once again we arrive at a contradiction.

REFERENCES

1. S. A. Amitsur, *On the semi-simplicity of group algebras*, Michigan Math. J. 6 (1959), 251-253.

2. G. Higman, *A finitely generated infinite simple group*, J. London Math. Soc. 26 (1951), 61-64.

3. A. G. Kurosh, *The theory of groups*, Vol. 2, Chelsea Publishing Co., New York, N. Y. (1956).

4. D. S. Passman, *Nil ideals in group rings*, Michigan Math. J. 9 (1962), 375-384.

5. O. E. Villamayor, *On the semisimplicity of group algebras*, Proc. Amer. Math. Soc. 9 (1958), 621-627.

Brandeis University