INVOLUTIONS FIXING PROJECTIVE SPACES

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The object of this paper is to prove the following result.

**Theorem.** Suppose \((T, M^n)\) is a differentiable involution on a closed manifold \(M^n (n > 2r)\), and its fixed point set is real projective space \(\text{RP}(2r)\). Then \(n = 4r\), and \((T, M^n)\) is cobordant to the involution of \(\text{RP}(2r) \times \text{RP}(2r)\) that sends \((x, y)\) into \((y, x)\).

This result was suggested by Conner and Floyd [2, Section 27]. In particular, Conner and Floyd proved that \(n = 4r\), and that if \(\xi: E \to \text{RP}(2r)\) denotes the normal bundle of \(\text{RP}(2r)\) in \(M^n\), then the Stiefel-Whitney class of \(\xi\) is \((1 + d)^m\), where both \(m\) and the binomial coefficient \(\binom{m}{2r}\) are odd, and where \(d\) is the nonzero class of \(H^1(\text{RP}(2r); \mathbb{Z}_2)\).

**Proof of the theorem.** Let \(\text{RP}(\xi)\) be the total space of the \(\text{RP}(2r - 1)\)-bundle associated with \(\xi\), and let \(p: \text{RP}(\xi) \to \text{RP}(2r)\) be the projection. Borel and Hirzebruch [1] have shown that \(H^*(\text{RP}(\xi); \mathbb{Z}_2)\) is the free module over \(H^*(\text{RP}(2r); \mathbb{Z}_2)\), via \(p^*\), on the classes \(1, c, \ldots, c^{2r-1}\), where \(c\) is the characteristic class of the double cover of \(\text{RP}(\xi)\) by the sphere bundle of \(\xi\). Multiplication in \(H^*(\text{RP}(\xi); \mathbb{Z}_2)\) is given by the formula

\[
0 = \sum_{i=0}^{2r} c^{2r-i} p^*(w_i(\xi)) = \sum_{i=0}^{2r} \binom{m}{i} c^{2r-i} \alpha^i
\]

\[
= c^{2r} + c^{2r-1} \alpha + \text{terms of higher degree in } \alpha
\]

(since \(m\) is odd), where \(\alpha = p^*(d)\). The Stiefel-Whitney class of \(\text{RP}(\xi)\) is

\[
w = (1 + \alpha)^{2r+1} \left\{ \sum_{i=0}^{2r} \binom{m}{i} (1 + c)^{2r-i} \alpha^i \right\}.
\]

(See [2, Theorem 23.3].)

By Theorem 28.1 of [2], the antipodal involution on the sphere bundle of \(\xi\) bounds a free involution, or equivalently, all of the generalized Stiefel-Whitney numbers \(c^i w_\omega[\text{RP}(\xi)]\) of \(\text{RP}(\xi)\) are zero (here \(w_\omega\) denotes any product \(w_{i_1} \cdots w_{i_s}\) of Stiefel-Whitney classes).

Since \(m\) and \(\binom{m}{2r}\) are odd, \(m \geq 2r + 1\). If \(m = 2r + 1\), then the bundle \(\xi\) and the normal bundle of \(\text{RP}(2r)\) in \(\text{RP}(2r) \times \text{RP}(2r)\), which is the tangent bundle \(\tau\) of \(\text{RP}(2r)\), have the same Stiefel-Whitney class. Thus the bundles \((\xi, \text{RP}(2r))\) and

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\((\tau, \text{RP}(2r))\) are cobordant, so that the normal bundles of the fixed point sets of the two involutions are cobordant. By Theorem 28.1 of [2], the involutions are then cobordant.

Thus, one may assume that \(m > 2r + 1\). Since \(\alpha^i = 0\) for \(i > 2r\),

\[
\begin{align*}
\nu &= \frac{(1 + \alpha)^{2r+1}}{(1 + c)^{m-2r} (1 + c + \alpha)^m} \\
&\quad \left(1 + c + \alpha\right)^{m-2r} \sum_{i=0}^{m-2r} \binom{m}{i} (1 + c)^{m-i} \alpha^i \\
&= \frac{(1 + \alpha)^{2r+1}}{(1 + c)^{m-2r} (1 + c + \alpha)^m} \left(1 + \alpha\right)^{2r+1} \\
&\quad \left(1 + c + \alpha\right)^{m-(2r+1)} \\
&= \frac{(1 + c + \alpha(c + \alpha))^{2r+1}}{(1 + c)^{m-(2r+1)}} \\
&= \left\{1 + \alpha(c + \alpha) + ca(c + \alpha) + c^2 \alpha(c + \alpha) + \cdots\right\} \cdot \left\{1 + c^2 + \alpha^2(c + \alpha)^2\right\}^{2r+1}.
\end{align*}
\]

Because \(m\) is odd, \(m - (2r + 1)\) is even. Let \(m - (2r + 1) = 2^s(1 + 2v)\), with \(v \geq 0\), \(s \geq 1\).

If \(s = 1\), then \(w_2(\text{RP}(\xi)) = \alpha(c + \alpha) + rc^2 + \alpha^2\) or \(ca = w_2(\text{RP}(\xi)) + rc^2\). Then

\[
c^{2r} \alpha^{2r-1} = c(c\alpha)^{2r-1} = c(w_2 + rc^2)^{2r-1},
\]

which is zero, since it gives a generalized Stiefel-Whitney number when evaluated on the fundamental class of \(\text{RP}(\xi)\). On the other hand,

\[
c^{2r} = c^{2r-1} \alpha + \text{terms of higher degree in } \alpha,
\]

so that

\[
c^{2r} \alpha^{2r-1} = c^{2r-1} \alpha^{2r},
\]

which is the nonzero class of \(H^{4r-1}(\text{RP}(\xi); \mathbb{Z}_2)\).

Thus \(s > 1\), and \(w_2(\text{RP}(\xi)) = \alpha(c + \alpha) + rc^2\) or \(\alpha(c + \alpha) = w_2 + rc\).

If \(2^s \leq 2r\), then

\[
w(\text{RP}(\xi)) = \frac{(1 + c + w_2 + rc^2)^{2r+1}}{(1 + c)} \left\{1 + \alpha^{2^s} + \cdots\right\},
\]

so that

\[
w_{2s}(\text{RP}(\xi)) = \alpha^{2^s} + P(w_2, c),
\]

where \(P(w_2, c)\) denotes a polynomial in \(w_2\) and \(c\). Since \(2r \geq 2^s\), we let \(2r = k2^s + u\) \((k \geq 1, \ 0 \leq u < 2^s)\). Then

\[
[\alpha(c + \alpha)]^u \alpha^{k2^s} c^{2r-1-u} = \alpha^u \alpha^{k2^s} \alpha^{2r-1-u} = \alpha^{2r} \alpha^{2r-1},
\]

because all other terms of \([\alpha(c + \alpha)]^u \alpha^{k2^s}\) contain a higher power of \(\alpha\) and are therefore zero. Thus

\[
c^{2r-1} \alpha^{2r} = (w_2 + rc^2)^u \left(w_2 + P(w_2, c)\right)^k c^{2r-1-u}
\]
is zero, since it gives a generalized Stiefel-Whitney number. Since this is still the nonzero class, we have established a contradiction.

Thus $2^s > 2r$, and

$$w(\xi) = (1 + d)^m = (1 + d)^{2r+1} (1 + d)^{m-(2r+1)} = (1 + d)^{2r+1} (1 + d^{2^s})^{2r+1};$$

but $d^{2^s} = 0$, since $2^s > 2r$. Hence $w(\xi) = (1 + d)^{2r+1}$, and as we noted for the case $m = 2r + 1$, this establishes the theorem.

REFERENCES

