ON THE MONOTONICITY OF THE ZEROS OF TWO POWER SERIES

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1. In the preceding paper, Peyerimhoff considers the functions f_κ and g_κ defined by the equations

$$f_\kappa(z) = \sum_{n=0}^{\infty} (n+1)^\kappa z^n \quad \text{and} \quad g_\kappa(z) = \sum_{n=0}^{\infty} (1 - c^{n+1})^\kappa z^n \quad (0 < c < 1)$$

in the unit circle, and by analytic continuation in C^*, the complex plane with a cut from 1 to ∞ along the positive real axis. In particular, he shows that these functions have exactly k zeros ($k < \kappa \leq k+1$) in C^*, and that the zeros are all negative and simple. His proof, as well as certain numerical calculations, indicate that the zeros are monotone functions of κ. For the sake of a better understanding of the functions f_κ and g_κ, it seems worthwhile to investigate this question. A proof that the two zeros of f_κ nearest to the origin and the first zero of g_κ are monotonic was communicated to me by A. Peyerimhoff. In this paper we shall show that *all zeros of* f_κ *and* g_κ *are monotonically increasing functions of* κ.

2. Let us consider f_κ first. We denote the zeros by $\xi_i(\kappa)$, with

$$0 > \xi_1(\kappa) > \cdots > \xi_k(\kappa).$$

From the paper of Peyerimhoff we take the relation

$$\xi_{i+1}(\kappa+1) < \xi_i(\kappa) < \xi_i(\kappa+1) \qquad (1 \leq i < \kappa). \tag{1}$$

Since the $\xi_i(\kappa)$ are simple and $f_\kappa(0) = 1 > 0$,

$$\operatorname{sgn} f'_\kappa(\xi_i(\kappa)) = (-1)^{i-1}.$$

From the relation

$$\frac{d\xi_i(\kappa)}{d\kappa} = -\frac{\partial f_\kappa(\xi_i(\kappa))}{d\kappa} \Big/ f'_\kappa(\xi_i(\kappa))$$

it will follow that $\frac{d}{d\kappa}\xi_i(\kappa) > 0$, when we have shown that

$$\operatorname{sgn} \frac{\partial f_\kappa(\xi_i(\kappa))}{d\kappa} = (-1)^i. \tag{2}$$

From the definition of f_κ we see that

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$$\frac{\partial f_\kappa(z)}{\partial \kappa} = \sum_{n=1}^{\infty} (n+1)^\kappa z^n \log(n+1).$$

Since

$$\int_0^1 \frac{(1-t^n)\,dt}{\log 1/t} = \int_0^1 \int_0^n t^x \,dx\,dt = \int_0^n \int_0^1 t^x \,dt\,dx = \int_0^n \frac{dx}{1+x} = \log(1+n),$$

this implies that

$$\frac{\partial f_\kappa(z)}{\partial \kappa} = \int_0^1 \frac{(f_\kappa(z) - f_\kappa(tz))\,dt}{\log 1/t}. \tag{3}$$

So far the formula is proved in the unit disk only, but analytic continuation shows immediately that it holds in the region C^*. We shall need it only for $z < 0$. If we replace z by $\xi_i(\kappa)$, (3) takes the simpler form

$$\frac{\partial f_\kappa(\xi_i(\kappa))}{\partial \kappa} = -\int_0^1 \frac{f_\kappa(t\,\xi_i(\kappa))\,dt}{\log 1/t} = \frac{-1}{|\xi_i(\kappa)|} \int_{\xi_i(\kappa)}^0 \frac{f_\kappa(x)\,dx}{\log \xi_i(\kappa)/x}.$$

We split this integral into two integrals from $\xi_i(\kappa)$ to $\xi_{i-1}(\kappa - 1)$ (the latter value lies to the right of $\xi_i(\kappa)$, because of (1)) and from $\xi_{i-1}(\kappa - 1)$ to 0. Partial integration of this last integral, with the help of the relation $f_\kappa(z) = (z f_{\kappa-1}(z))'$, yields the equations

$$\int_{\xi_{i-1}(\kappa-1)}^0 \frac{f_\kappa(x)\,dx}{\log \xi_i(\kappa)/x}$$

$$= -\frac{\xi_{i-1}(\kappa-1) f_{\kappa-1}(\xi_{i-1}(\kappa-1))}{\log \xi_i(\kappa)/\xi_{i-1}(\kappa-1)} - \int_{\xi_{i-1}(\kappa-1)}^0 \frac{f_{\kappa-1}(x)\,dx}{[\log \xi_i(\kappa)/x]^2}$$

$$= -\int_{\xi_{i-1}(\kappa-1)}^0 \frac{f_{\kappa-1}(x)\,dx}{[\log \xi_i(\kappa)/x]^2}.$$

We continue in this manner; that is, we split the last integral into two integrals from $\xi_{i-1}(\kappa - 1)$ to $\xi_{i-2}(\kappa - 2)$ and from $\xi_{i-2}(\kappa - 2)$ to 0, and integrate the latter integral by parts, and so forth. This procedure leads to the formula

$$\frac{\partial f_\kappa(\xi_i(\kappa))}{\partial \kappa} = \frac{1}{|\xi_i(\kappa)|} \sum_{j=1}^{i} (-1)^j (j-1)! \int_{\xi_{i-j+1}(\kappa-j+1)}^{\xi_{i-j}(\kappa-j)} \frac{f_{\kappa-j+1}(x)\,dx}{[\log \xi_i(\kappa)/x]^j}, \tag{4}$$

where $\xi_0(\kappa - i)$ stands for 0.

Now, since $f_\kappa(0) = 1 > 0$ and f_κ changes sign at each zero, it follows from (1) that

$$(\xi_{i-j+1}(\kappa - j + 1), \xi_{i-j}(\kappa - j)) \subseteq (\xi_{i-j+1}(\kappa - j + 1), \xi_{i-j}(\kappa - j + 1)),$$

and therefore

$$\operatorname{sgn} f_{\kappa-j+1}(x) = (-1)^{i-j} \quad \text{in the interval } (\xi_{i-j+1}(\kappa - j + 1),\ \xi_{i-j}(\kappa - j)).$$

From (4) we now obtain (2) and thereby our proposition.

3. We now pass to g_κ. The proof is similar to the proof in Section 2, but slightly more complicated because in the summation by parts that follows, the analogue to the term that vanished in Section 2 is not zero.

Let $\eta_i(\kappa)$ denote the zeros of g_κ, with

$$0 > \eta_1(\kappa) > \eta_2(\kappa) > \cdots > \eta_k(\kappa) \qquad (k < \kappa \leq k+1).$$

Peyerimhoff's proof of the existence of k zeros of g_κ shows that

$$c\,\eta_{i+1}(\kappa + 1) < \eta_i(\kappa) < \eta_i(\kappa + 1) \qquad (i = 1, 2, \cdots, k). \tag{5}$$

The problem is again to show that

$$\operatorname{sgn} \frac{\partial g_\kappa(\eta_i(\kappa))}{\partial \kappa} = (-1)^i. \tag{6}$$

For this purpose we choose the integers n_j $(j = 1, 2, \cdots, i-1)$ subject to the condition

$$c^{n_j}\eta \leq \eta_{i-j}(\kappa - j) < c^{n_j+1}\eta \qquad (\eta = \eta_i(\kappa)), \tag{7}$$

and we take $n_0 = 0$ and $n_i = \infty$. By (5), $n_0 < n_1 < \cdots < n_{i-1}$.

Expanding $\log(1 - c^{n+1})$ into a Taylor series and changing the order of summation in the formula

$$\frac{\partial g_\kappa(z)}{\partial \kappa} = \sum_{n=0}^{\infty} (1 - c^{n+1})^\kappa z^n \log(1 - c^{n+1}),$$

we find that

$$\frac{\partial g_\kappa(z)}{\partial \kappa} - \sum_{\nu=1}^{\infty} \frac{c^\nu}{\nu} g_\kappa(c^\nu z). \tag{8}$$

Like (3), this formula holds not only for $|z| < 1$, but for all $z \in C^*$.

In (8), with $z = \eta_i(\kappa) = \eta$, and using the relation $g_\kappa(z) = g_{\kappa-1}(z) - c g_{\kappa-1}(cz)$, we sum by parts from n_1 to ∞:

$$\sum_{\nu=n_1}^{\infty} \frac{c^\nu}{\nu} g_\kappa(c^\nu \eta) = \sum_{\nu=n_1}^{\infty} \frac{c^\nu}{\nu} (g_{\kappa-1}(c^\nu \eta) - c g_{\kappa-1}(c^{\nu+1}\eta))$$

$$= \frac{c^{n_1}}{n_1} g_{\kappa-1}(c^{n_1}\eta) - \sum_{\nu=n_1+1}^{\infty} \frac{c^\nu}{\nu(\nu-1)} g_{\kappa-1}(c^\nu \eta).$$

More generally, for $j = 1, 2, \cdots, i - 1$ we get the formula

$$\sum_{\nu=n_j}^{\infty} \frac{(j-1)!\, c^{\nu}}{\nu(\nu-1)\cdots(\nu-j+1)}\, g_{\kappa-j+1}(c^{\nu}\eta) \tag{9}$$
$$= \frac{(j-1)!\, c^{n_j}}{n_j(n_j-1)\cdots(n_j-j+1)}\, g_{\kappa-j}(c^{n_j}\eta) - \sum_{\nu=n_j+1}^{\infty} \frac{j!\, c^{\nu}}{\nu(\nu-1)\cdots(\nu-j)}\, g_{\kappa-j}(c^{\nu}\eta).$$

Repeated application of (9) to (8) leads to the equation

$$\frac{\partial g_{\kappa}(\eta)}{\partial \kappa} = \sum_{j=1}^{i} (-1)^j \sum_{\nu=n_{j-1}+1}^{n_j-1} \frac{(j-1)!\, c^{\nu}}{\nu(\nu-1)\cdots(\nu-j+1)}\, g_{\kappa-j+1}(c^{\nu}\eta) \tag{10}$$
$$+ \sum_{j=1}^{i-1} \frac{(-1)^j (j-1)!\, c^{n_j}}{n_j(n_j-1)\cdots(n_j-j+1)}\, g_{\kappa-j}(c^{n_j}\eta).$$

For $n_{j-1} < \nu < n_j$, it follows from (5) and (8) that

$$c^{\nu}\eta < c^{n_j}\eta \le \eta_{i-j}(\kappa - j) < \eta_{i-j}(\kappa - j + 1),$$

$$c^{\nu}\eta \ge c^{n_{j-1}+1}\eta > \eta_{i-j+1}(\kappa - j + 1).$$

Since $g_{\lambda}(0) = 1 > 0$ and g_{κ} changes sign at each zero, we see that

$$\operatorname{sgn} g_{\kappa-j+1}(c^{\nu}\eta) = (-1)^{i-j} \quad \text{if } n_{j-1} < \nu < n_j. \tag{11}$$

Similarly, the inequalities

$$\eta_{i-j+1}(\kappa - j) < c^{-1}\eta_{i-j}(\kappa - j) < c^{n_j}\eta \le \eta_{i-j}(\kappa - j)$$

imply that

$$\operatorname{sgn} g_{\kappa-j}(c^{n_j}\eta) = \begin{cases} (-1)^{i-j} \\ \text{or } 0. \end{cases} \tag{12}$$

Now (6) is an immediate consequence of (10), (11), and (12).

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