THE SUM OF TWO CRUMPLED CUBES

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A *crumpled cube* is a space that is homeomorphic to the closure of the interior of a 2-sphere in E^3. There exist many examples of crumpled cubes that are not 3-cells, the best known probably being the examples described by Alexander [1] and by Fox and Artin [7].

Suppose that C and D are crumpled cubes and h is a homeomorphism of Bd C onto Bd D. Let $C \cup_h D$ denote the space obtained by identification of C and D along their boundaries by the homeomorphism h.

Hosay [10] and Lininger [11] have independently shown that if C is a 3-cell, then $C \cup_h D$ is S^3. Bing has shown [4] that if each of C and D is the example of Alexander and h is the identity, then $C \cup_h D$ is S^3. It is known that if each of C and D is the example of Fox and Artin and h is the identity, then $C \cup_h D$ is not S^3.

The goal of this paper is to study certain conditions that are necessary in order that $C \cup_h D$ be S^3.

Suppose that K is a crumpled cube and p is a point of Bd K. The statement that p *is a piercing point of* K means that there exists an embedding $f\colon K \to S^3$ such that f(Bd K) can be pierced by a tame arc at f(p). If K is a 3-cell or the example of Alexander, then each point of Bd K is a piercing point of K. If K is the example of Fox and Artin, then K has exactly one nonpiercing point. Stallings [15] has given an example of a crumpled cube with uncountably many nonpiercing points.

The main result of this paper is the following theorem.

THEOREM. *Suppose that each of* C *and* D *is a crumpled cube,* h *is a homeomorphism of* Bd C *onto* Bd D, *and* $C \cup_h D$ *is* S^3. *Then, if* p *is a nonpiercing point of* C, h(p) *is a piercing point of* D.

We shall establish the theorem by using the theorem of Lininger and Hosay to view $C \cup_h D$ as a decomposition of S^3 into points and arcs. Lemma 6 will show that each arc in this decomposition is cellular, Lemma 4 will show that each of the arcs is locally tame except perhaps at *one* of its endpoints, and an application of Lemma 3 will establish the theorem.

LEMMA 1. *Suppose that* K *is a crumpled cube and* p *is a piercing point of* K. *Then, if* $f\colon K \to S^3$ *is an embedding such that* $\mathrm{Cl}(S^3 - f(K))$ *is a 3-cell,* f(Bd K) *can be pierced by a tame arc at* f(p).

Proof. Let $f\colon K \to S^3$ be an embedding such that $\mathrm{Cl}(S^3 - f(K))$ is a 3-cell, and let $g\colon K \to S^3$ be an embedding such that g(Bd K) can be pierced by a tame arc at g(p). It follows from a theorem of Gillman [8] that some tame arc α on g(Bd K) contains g(p).

The proof of Theorem 2 of [11] shows that there exists a homeomorphism $h\colon g(K) \to S^3$ such that (i) $\mathrm{Cl}(S^3 - hg(K))$ is a 3-cell and (ii) the restriction of h to α is the identity. Now, since each of $\mathrm{Cl}(S^3 - f(K))$ and $\mathrm{Cl}(S^3 - hg(K))$ is a 3-cell, the homeomorphism hgf^{-1} can be extended to a homeomorphism of S^3 onto itself

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that takes f(K) onto hg(K). Then $fg^{-1}(\alpha)$ is a tame arc on f(Bd K) that contains f(p), and it follows from [8] that f(Bd K) can be pierced by a tame arc at f(p). This establishes Lemma 1.

Let B denote the unit 3-cell in E^3. If $x \in$ Bd B, let α_x denote the straight-line interval from the origin to x.

LEMMA 2. *Let* $h: B \to E^3$ *be an embedding. Then* h(Bd B) *can be pierced by a tame arc at* h(p) *if and only if* $h(\alpha_p)$ *is tame.*

Proof. Let p be a point of Bd B. Suppose first that $h(\alpha_p)$ is tame. Let β be an arc on h(Bd B) that has h(p) as an endpoint and is locally tame except perhaps at h(p). Let C denote the cone from the origin over $h^{-1}(\beta)$, and let D denote h(C). Then D is a 2-cell and is locally tame except perhaps at h(p). Now the arc $h(\alpha_p)$ is tame, contains h(p), and lies on D. It follows from a theorem of Doyle and Hocking [6] that D is tame. Hence β is tame, and it follows from [8] that h(Bd B) can be pierced by a tame arc at h(p).

Now suppose that h(Bd β) can be pierced by a tame arc at h(p). By [8], some tame arc β on h(Bd B) contains h(p). Let C be the cone from the origin over $h^{-1}(\beta)$, and let D be h(C). Then D is locally tame except perhaps at h(p), and h(p) lies on the tame arc β. As before, D is tame, and it follows that $h(\alpha_p)$ is tame. This establishes Lemma 2.

Lemmas 1 and 2 combine to yield Lemma 3.

LEMMA 3. *Let* $h: B \to S^3$ *be an embedding. Then* h(p) *is a piercing point of* $Cl(S^3 - h(B))$ *if and only if* $h(\alpha_p)$ *is tame.*

Suppose that X is a subset of S^3. The statement that X is *cellular* means that there exists a sequence $C_1, C_2, \cdots$ of 3-cells such that (i) $C_{i+1} \subset \text{Int } C_i$ and (ii) $X = \bigcap_{i=1}^{\infty} C_i$.

If G is an upper-semicontinuous decomposition of S^3, we denote by S^3/G the resulting decomposition space, and by P the natural projection map of S^3 onto S^3/G. If X is a cellular arc of a cellular 3-cell in S^3, we denote by S^3/X the decomposition space obtained from the decomposition whose only nondegenerate element is X.

The following lemma is the result of joint work by David Gillman and the author. The author thanks the referee for simplifying the original proof.

LEMMA 4. *Suppose that* α *is a cellular arc in* S^3. *Then* α *cannot fail to be locally tame at exactly its two endpoints.*

Proof. Suppose that α is a cellular arc in S^3 with endpoints a and b. Suppose further that α is locally tame at each point of $\alpha - \{a, b\}$ and that α fails to be locally tame at each of a and b.

Let q be a point of $\alpha - \{a, b\}$, and suppose that D is a tame disk that intersects α in exactly the point q, and that this disk is pierced by α at q. Now S^3/α is S^3, and it follows from [5] that there exist a subdisk D' of P(D) and a tame 2-sphere S' such that $P(q) \in \text{Int } D'$ and $D' \subset S'$. Let S denote the 2-sphere $P^{-1}(S')$. Then S is tame, intersects α in exactly the point q, and is pierced by α at q.

Now there is a 3-cell C such that α lies on the boundary of C, $S \cap C$ is a spanning disk of Bd C, and C is locally tame except at the points a and b. Let C_1 and C_2 denote the closures of the components of C - S containing a and b, respectively. Since S is tame, there exist 3-cells K_1 and K_2 such that (i) $K_1 \cap K_2 = S$, and (ii) $C_1 \subset K_1$ and $C_2 \subset K_2$.

Because each of C_1 and C_2 is wild, neither $Cl(K_1 - C_1)$ nor $Cl(K_2 - C_2)$ is a 3-cell. Since α is cellular, C is cellular, and hence S^3/C is S^3. The 2-sphere P(S) is locally tame except at one point, and it follows from [9] that the closure of one of the complementary domains of P(S) is a 3-cell. Then one of $K_1 - C_1$ and $K_2 - C_2$ is topologically $E^2 \times [0, 1)$, and it follows that one of $Cl(K_1 - C_1)$ and $Cl(K_2 - C_2)$ is a 3-cell. This contradiction establishes Lemma 4.

LEMMA 5. *Suppose that* G *is an upper-semicontinuous decomposition of* S^3, *each nondegenerate element of* G *is a compact absolute retract, and* U *is a simply connected open set in* S^3/G. *Then* $P^{-1}(U)$ *is simply connected.*

Lemma 5 is a consequence of Smale's Vietoris mapping theorem for homotopy [14]. Nevertheless, we include a proof, for completeness.

Proof. Let $g \in G$. Then, since g is a compact absolute retract, there exists a sequence $V_1, V_2, \cdots$ of open sets such that $\bigcap_{i=1}^{\infty} V_i = g$ and such that, for each i, $V_{i+1} \subset V_i$ and V_{i+1} is homotopic to 0 in V_i. Furthermore, since G is upper-semicontinuous, we may assume that for each i, V_i is the union of elements of G.

Now suppose that U is a simply connected open set in S^3/G. For each element g of G such that $g \subset P^{-1}(U)$, let W_g and Y_g be open sets such that $g \subset W_g$, $W_g \subset Y_g$, W_g is homotopic to 0 in Y_g, $Y_g \subset P^{-1}(U)$, and each of W_g and Y_g is the union of elements of G.

Let σ be a 2-simplex, and let f be a map of Bd σ into $P^{-1}(U)$. Let g be the map of Bd σ into U defined by $g = Pf$. Now, since U is simply connected, some extension g^* of g maps σ into U. Let C be a finite subcollection of

$$\{W_g: g \subset P^{-1}(U)\}$$

such that C covers $P^{-1}(g^*(\sigma))$. The existence of C follows from the fact that for each g, $P(W_g)$ is an open set.

Now let T be a triangulation of σ whose mesh is so small that if τ is a 2-simplex of T, then there exists an element W of C such that $P^{-1}(g^*(\tau)) \subset W$. For each 2-simplex τ of T, let W_τ be an element of C such that $P^{-1}(g^*(\tau)) \subset W_\tau$.

We are now ready to extend the map f. This will be done by first extending f to the 0-skeleton of T, then to the 1-skeleton, and finally to the 2-skeleton.

Suppose that v is a vertex of T, not in Bd σ. Then let P_v be a point of $P^{-1}(g^*(v))$, and set $f^*(v) = P_v$.

Suppose that $\langle v_1 v_2 \rangle$ is a 1-simplex in T that does not lie in Bd σ. Now $P^{-1}(g^*(\langle v_1 v_2 \rangle))$ is a compact connected set, and it follows that if τ and τ' are 2-simplexes in T and each has $\langle v_1 v_2 \rangle$ as a face, then P_{v_1} and P_{v_2} belong to the same component of $W_\tau \cap W_{\tau'}$. Let α be an arc in $W_\tau \cap W_{\tau'}$ from P_{v_1} to P_{v_2}, and let h be a homeomorphism of $\langle v_1 v_2 \rangle$ onto α such that $h(v_1) = P_{v_1}$. Then if $x \in \langle v_1 v_2 \rangle$, let $f^*(x)$ be $h(x)$.

Now suppose that τ is a 2-simplex in T. Then f^* is defined on Bd τ, and $f^*(\text{Bd } \tau) \subset W_\tau$. Since W_τ is homotopic to 0 in Y_τ, f^* may be extended to τ. This establishes Lemma 5.

LEMMA 6. *Suppose that* G *is an upper-semicontinuous decomposition of* S^3 *such that each element of* G *is a compact absolute retract, and such that* S^3/G *is a 3-manifold. Then each element of* G *is cellular.*

Proof. Let g be an element of G, and let V be an open set containing g. Let V' be an open set such that $V' \subset V$, $g \subset V'$, and V' is the union of elements of G. Then P(V') is open, and $P(g) \in P(V')$. Let B be an open 3-cell such that $P(g) \in B$ and $B \subset P(V')$. Then $B - \{p(g)\}$ is simply connected; it follows from Lemma 5 that $P^{-1}(B) - g$ is simply connected, and from McMillan's cellularity criterion [13] that g is cellular.

Proof of the Theorem. Suppose that each of C and D is a crumpled cube, h is a homeomorphism of Bd C onto Bd D, and $C \cup_h D$ is S^3. It follows from [11] that there exists an embedding f of $S^2 \times I$ in S^3 such that $Cl(S^3 - f(S^2 \times I))$ is $C \cup D$, and if y is a point of S^2, then the arc $f(\{y\} \times I)$ has endpoints x and h(x) on C and D, respectively. Now the space $C \cup_h D$ may be obtained by taking the decomposition of S^3 whose only nondegenerate elements are the sets $f(\{y\} \times I)$. Lemma 6, together with the hypothesis that $C \cup_h D$ is S^3, implies that each of the arcs $f(\{y\} \times I)$ is cellular. It follows from Lemma 4 that one of the arcs $f(\{y\} \times [0, 1/2])$ and $f(\{y\} \times [1/2, 1])$ is tame, and from Lemma 3 that if $f(\{y\} \times I)$ has endpoints x and h(x), then either x is a piercing point of C or h(x) is a piercing point of D. This establishes the theorem.

We point out that the conditions stated in our theorem, while necessary for $C \cup_h D$ to be S^3, are not sufficient. Ball [3] has described a crumpled cube C each of whose boundary points is a piercing point, and a homeomorphism h: Bd C $\to$ Bd C such that $C \cup_h C$ is not S^3.

We can use Lemma 6 and a theorem of Armentrout to obtain a proof of a theorem announced by Lininger [12].

THEOREM (Lininger). *If* C *and* D *are crumpled cubes,* h: Bd C $\to$ Bd D *is a homeomorphism, and* $C \cup_h D$ *is a* 3*-manifold, then* $C \cup_h D$ *is* S^3.

Proof. As in the proof of the theorem above, we represent $C \cup_h D$ as a decomposition of S^3 whose only nondegenerate elements are the fibering arcs of an annulus. It follows from Lemma 6 that each of the fibering arcs is cellular. Since some open set in S^3 misses the sum of the nondegenerate elements, it follows from [2] that the decomposition space is S^3.

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