EXTREME POINTS OF THE NUMERICAL RANGE OF A HYPONORMAL OPERATOR

J. G. Stampfli

We denote the numerical range of an operator T by $W(T) = \{(Tx, x); \|x\| = 1\}$.

In [1], Donoghue lists the most important facts about $W(T)$. There are six of these, and four place no special restriction on the operator. We list the remaining two:

a) If T is normal, then the closure of $W(T)$ is the smallest convex set containing the spectrum of T.

b) If T is normal and $W(T)$ is closed, the extreme points of $W(T)$ are eigenvalues.

It has been shown independently by Putnam [2] and Stampfli [4] that a) holds when normality is replaced by hyponormality. It is the purpose of this note to show that the same replacement is valid in b).

An operator T is said to be hyponormal if $T^*T - TT^* \geq 0$, or equivalently, $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. Throughout, H will denote the underlying Hilbert space.

LEMMA 1. *Let* $\Re W(T) \geq 0$. *Then* $(Tx, x) = 0$ *implies* $Tx = -T^*x$.

Proof. If $x \in H$, then $((T + T^*)x, x) = 2\Re(Tx, x) \geq 0$ or $T + T^* \geq 0$. Now, $(Tx, x) = 0$ implies $((T + T^*)x, x) = 0$, and thus $(T + T^*)x = 0$ since $T + T^*$ is positive.

LEMMA 2. *Let* $\Re W(T) \geq 0$, *and let* 0 *be an extreme point of* $W(T)$. *Then* $M = \{x \in H: (Tx, x) = 0\}$ *is a closed subspace.*

Proof. All is clear but the linearity. For $x, y \in M$, we see that

$$\begin{aligned}(T(x + y), (x + y)) &= (Tx, x) + (Tx, y) + (Ty, x) + (Ty, y)\\ &= (Tx, y) + \overline{(T^*x, y)} = (Tx, y) - \overline{(Tx, y)}\\ &= 2\,\Im(Tx, y) = a.\end{aligned}$$

Assume $a \neq 0$. Then $(T(e^{i\theta}x + y), (e^{i\theta}x + y)) = 2\,\Im\, e^{i\theta}(Tx, y)$. Now, since $a \neq 0$, for $e^{i\theta} = \pm 1$ the values of $2\,\Im\, e^{i\theta}(Tx, y)$ lie in both the upper and lower half-planes. Thus 0 is not an extreme point, contrary to hypothesis.

If we define $N = \{x \in H: Tx = -T^*x\}$, then clearly N is a closed subspace. From Lemma 1 we see that $M \subset N$ when $\Re W(T) \geq 0$.

We shall need the following result on hyponormal operators. (See Lemma 3 of [4]).

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THEOREM A. *Let* T *be hyponormal; then* $\|Tx\| = \|T^*x\|$ *if and only if* $TT^*x = T^*Tx$.

LEMMA 3. *Let* T *be hyponormal, and for a fixed real value* θ, *let*

$$K = \{x: x \in H;\ Tx = e^{i\theta} T^* x\}.$$

Then K *is a reducing subspace of* T, *and* $T \mid K$ *is normal. Thus* $T^n x = e^{in\theta} T^{*n} x$ *for* $x \in K$.

Proof. For $x \in K$, $Tx = e^{i\theta} T^* x$ implies $\|Tx\| = \|T^*x\|$ and hence $TT^*x = T^*Tx$. But then

$$T(Tx) = T(e^{i\theta} T^* x) = e^{i\theta} T^*(Tx) \quad \text{and} \quad T(T^*x) = T^*(Tx) = e^{i\theta} T^*(T^*x),$$

for $x \in K$. From this we may conclude that K is invariant under both T and T^*, in other words, that it is reducing. Since $T^*Tx = TT^*x$ for $x \in K$, it follows that $T \mid K$ is normal. The last statement of the theorem is clear from the definition of K, its invariance under T, and the normality of $T \mid K$.

COROLLARY 1. *If* T *is hyponormal and* N *is defined as above, then* N *is a reducing subspace of* T, *and* $T \mid N$ *is normal.*

Proof. Take $\theta = \pi$.

COROLLARY 2. *If* T *is hyponormal and* $Tx = e^{i\theta} T^* x$ *for some* $x \in H$, *then* T *has a proper invariant subspace.*

THEOREM 1. *Let* T *be hyponormal and* $\Re W(T) \geq 0$, *where* 0 *is an extreme point of* W(T). *If* $(Tx, x) = 0$, *then* $Tx = 0$. *Moreover,* $M = \{x \in H: (Tx, x) = 0\}$ *is a reducing subspace of* T.

Proof. We shall first give a quick proof by using the known result for normal operators. Then we shall employ a second method that makes no recourse to normality.

Since $(Tx, x) = 0$, we see that $x \in M \subset N$. Now T is normal on N, and therefore, under our hypothesis on W(T), the condition $(Tx, x) = 0$ implies that $Tx = 0$.

For a more barbaric approach, let $Tx = ay$, where $\|x\| = \|y\| = 1$ and $a \neq 0$. Since $(Tx, x) = 0$, it follows that $(x, y) = 0$. Thus

$$(Ty, x) = (y, T^*x) = \overline{-(Tx, y)} = \overline{-a}.$$

Set $L = \{x, y\}$, and define P to be the projection of H on L. Then the matrix representation of PTP on L is $\begin{vmatrix} 0 & -\bar{a} \\ a & c \end{vmatrix}$.

If $x \in N$, then $ay = Tx \in N$, hence $(Ty, y) = \overline{-(Ty, y)}$ and c is pure imaginary. Observe that $W(PTP) \subset W(T)$. However, a calculation reveals that W(PTP) is the line segment joining the roots of the equation $\lambda^2 - c\lambda + |a|^2 = 0$ (for the matrix is normal). Now the roots are

$$\lambda = (c \pm i\sqrt{|c|^2 + 4|a|^2})/2.$$

Since 0 is an extreme point of W(T) and thus of W(PTP), it must be an endpoint of the line segment, that is, one of the roots. Clearly this happens only if $a = 0$, which implies $Tx = 0$. The last statement of the theorem is just a general fact about the eigenvalues of hyponormal operator (they reduce; see [3]).

COROLLARY 1. *If* T *is hyponormal and* z_0 *is an extreme point of* W(T), *then* $(Tx, x) = z_0$ *together with* $\|x\| = 1$ *implies that* $Tx = z_0 x$ *and* $\{x \in H\colon Tx = z_0 x\}$ *is a nonempty subspace.*

COROLLARY 2. *Let* T *be hyponormal on a separable Hilbert space* H. *Then* W(T) *has at most a countable number of extreme points.*

Proof. To every extreme point there corresponds an eigenvalue, and from [3] we know that distinct eigenvalues engender orthogonal eigenvectors. Since the space H is separable, we are done.

We remark that, in general, the extreme points of the numerical range of an arbitrary operator are not eigenvalues. One need look no further than that traditional stamping ground for counterexamples, the two-dimensional Hilbert space. For the operator

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix},$$

the numerical range is the closed disc $|z| \leq 1/2$. Thus every boundary point is an extreme point, though hardly an eigenvalue.

We note, however, that Donoghue has proved the following:

THEOREM. *If* z_0 *is an extreme point of* W(T) *and the boundary of* W(T) *is not differentiable at* z_0 *(roughly,* W(T) *has a corner at* z_0), *then* z_0 *is an eigenvalue.*

REFERENCES

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The University of Michigan
Ann Arbor, Michigan