PROPERTIES INHERITED BY RING EXTENSIONS

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Two of the basic methods of embedding a ring in a larger ring are the adjunction of an identity and the formation of the direct sum with another ring. They are special cases of the general ring extension defined as follows.

Definition 1. Let A and S be two rings. We say that a ring E is an *extension of* A *by* S if A is an ideal in E and E/A is isomorphic to S.

The more precise homological definition is not needed in this paper. Several authors have discussed whether a ring with a certain property can always be embedded in a ring with identity having the same property (see, for instance, the introduction to [5]). Similar problems have also been considered for general extensions, for example, in [8]. The present paper answers several questions of this type. The theorem in Section 2 is concerned with the special case of embedding a ring in a ring with identity, while the main body of results (Section 3) deals with general extensions. The two basic theorems, which yield several corollaries, concern semisimple rings. Sample corollary: Any extension of a Boolean ring by a Boolean ring is Boolean. Finally, we improve a theorem of Szendrei on the general extension of rings to rings with identity.

1. PRELIMINARIES

The problem of describing all extensions of A by S of the type defined above was solved by Everett [3], and the solution was later recast by Rédei [7]. (In some of the literature the term "Schreier extension" is used for Everett's extension.) In Rédei's formulation, each extension of A by S is viewed as a "skew product," that is, as the set $A \times S$ with addition and multiplication defined by

$$(a, s) + (b, t) = (a + b + [s, t], s + t),$$

$$(a, s)(b, t) = (ab + sb + at + \{s, t\}, st),$$

where $[s, t]$, $\{s, t\}$, sb, and at describe functions from $S \times S$, $S \times A$, and $A \times S$ into A. This is a ring if and only if a specific but rather long collection of identities involving these functions is valid. In the two classical cases mentioned in the introduction, the first two functions are, of course, the zero function. Further details, which may be found in [8], will not be given here; in fact, in most of the paper we do not use an explicit description of the extensions of A by S.

We shall now give the definitions of some of the less familiar concepts used in the sequel. The *characteristic* of a ring A is the common additive order of all the elements of A, if it exists; this is necessarily either a prime or infinity. (Our use of the word "characteristic" in [5] is different.) For any prime p, A is a *p-ring* if it has characteristic p and $a^p = a$ for each $a \in A$. If m is a positive integer, a ring A is *m-regular* if for each $a \in A$ there exists an $x \in A$ such that

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$a^m x a^m = a^m$; a *π-regular* ring is defined similarly, with the m allowed to vary, and a 1-regular ring is called *regular*.

The ring of integers modulo n will be denoted by I_n.

2. IDENTITY-ADJUNCTION EXTENSIONS

Assume that S is a commutative ring with identity over which A is an algebra. Then one extension of A by S is defined by the formulas

$$(a, s) + (b, t) = (a + b, s + t), \qquad (a, s)(b, t) = (ab + sb + ta, st);$$

that is, the functions sb and at are defined to be multiplication of b and a by the operators s and t, respectively. (Observe that both s and t are now written on the left.) This extension will be denoted by (A ; S). Since (A ; S) has an identity, and adjoining an identity by any of the usual methods yields an extension of this type, we call (A ; S) an *identity-adjunction extension.*

In [5], we considered the question whether (A ; S) is regular, m-regular, or π-regular if both A and S are. In the case of regularity, the answer is easily obtained; see Theorem 2 below. A more involved problem is whether there always exists a commutative regular ring S with identity over which a regular ring A is an algebra. Fuchs and Halperin [2] have, in fact, exhibited a regular ring S over which *every* regular ring is an algebra. As was pointed out in the review of [5] (Mathematical Reviews 18 (1957), p. 375), the proofs therein actually yield only the results that if A is commutative regular and S is m-regular (π-regular), then (A ; S) is m-regular (π-regular). (The statement in [6] that semisimplicity can be avoided in Lemma 5 of [5] unfortunately remains in doubt.) More than this can be obtained in a simpler fashion, however; see Theorem 2 below. The only result of this type that we have been able to obtain for a generalized regular ring A is the following.

THEOREM 1. *If* A *is a* p*-regular ring of characteristic* p, *where* p *is a prime, then* $(A ; I_p)$ *is a* p*-regular ring of characteristic* p.

Proof. Clearly, $(A ; I_p)$ is a ring of characteristic p. Note that

$$(a, s)^p = (a^p, s^p) = (a^p, s);$$

so, given (a, s), we are required to find (x, t) such that $(a^p, s)(x, t)(a^p, s) = (a^p, s)$. This reduces to solving the equation

$$a^p x a^p + s x a^p + s a^p x + s^2 x = a^p - t a^{2p} - 2 s t a^p, \tag{1}$$

where $sts = s$. We choose $t = 0$ if $s = 0$. Now

$$-ta^{2p} - a^p = -t^p a^{2p} - a^p = (-ta^2 - a)^p,$$

so there exists a $z \in A$ such that

$$(-ta^{2p} - a^p) z (-ta^{2p} - a^p) = -ta^{2p} - a^p.$$

Set $x = (st - 1) z + t^2 a^p z a^p$. Direct verification shows that this x is a solution of (1).

3. GENERAL EXTENSIONS

We now use an idea of Brown and McCoy to obtain some results about general extensions; see Lemma 1 and Theorem 2 of [1].

THEOREM 2. *If* A *is regular and* S *is* m*-regular* (π*-regular*), *then any extension* E *of* A *by* S *is* m*-regular* (π*-regular*).

Proof. The proof reads the same way in both cases, with m interpreted as variable in the π-regular case. Given $a \in E$, we can find $y \in E$ such that $a^m - a^m y a^m \in A$, because S is m-regular (π-regular). Thus, there exists $z \in A$ such that

$$(a^m - a^m y a^m) z (a^m - a^m y a^m) = a^m - a^m y a^m.$$

Set $x = z - za^m y - ya^m z + ya^m za^m y + y$; then $a^m x a^m = a^m$.

Varying this idea slightly, we have the following result, similar to both Theorems 1 and 2.

THEOREM 3. *If* A *is a* p*-regular ring, where* p *is a prime, and* S *is* m*-regular* (π*-regular*), *then any commutative extension* E *of* A *by* S, *of characteristic* p, *is an* mp*-regular* (π*-regular*) *ring.*

Proof. The proof reads the same way in both cases, with m interpreted as variable in the π-regular case. Given $a \in E$, we can find $y \in E$ such that $a^m - a^m y a^m \in A$, because S is m-regular (π-regular). Thus, there exists $z \in A$ such that

$$(a^m - a^m y a^m)^p z (a^m - a^m y a^m)^p = (a^m - a^m y a^m)^p.$$

Since E is commutative and of characteristic p, this can be written in the form

$$(a^{mp} - a^{mp} y^p a^{mp}) z (a^{mp} - a^{mp} y^p a^{mp}) = a^{mp} - a^{mp} y^p a^{mp}.$$

Set $x = z - za^{mp} y^p - y^p a^{mp} z + y^p a^{mp} za^{mp} y^p + y^p$; then $a^{mp} x a^{mp} = a^{mp}$.

Remark 1. The device used in Theorems 2 and 3 cannot be used to prove that E is m-regular if both A and S are m-regular (even if m is prime and E is commutative and of characteristic m). Indeed, the natural generalization of [1, Lemma 1] is false; that is, it is not true that if $a^m - a^m y a^m$ is m-regular, then a is m-regular (the definition of m-regularity of an element being the obvious one). For example, in the ring A of 2-by-2 matrices over I_4, define a to be the matrix of ones; then a is not 2-regular, but $a^2 - a^2 y a^2$ is 2-regular for every $y \in A$, since $a^2 y a^2 = 0$ for every $y \in A$ and $a^4 = 0$. (Note: this also shows that, in contrast with the regular case, the ring of matrices over an m-regular ring need not be m-regular.)

Remark 2. In Theorem 3, it is not sufficient to assume that A and S are commutative and of characteristic p—see Remarks 3 and 4 below.

The next theorem leads to a variety of results on properties inherited by ring extensions.

THEOREM 4. *Let* A *be isomorphic to a subdirect sum of the collection of primitive simple rings* $\{A_\alpha\}$, *and let* S *be isomorphic to a subdirect sum of the collection of primitive rings* $\{B_\beta\}$. *If* E *is any extension of* A *by* S, *then* E *is isomorphic to a subdirect sum of the collection of primitive rings* $\{A_\alpha\} \cup \{B_\beta\}$.

Proof. For each α and each β, let

$$M_\alpha = \{a \in A: a_\alpha = 0\} \quad \text{and} \quad P_\beta = \{s \in S: s_\beta = 0\},$$

where a_α and s_β denote components in the given subdirect sums. Then $\bigcap_\alpha M_\alpha = (0)$ and $\bigcap_\beta P_\beta = (0)$. Now the primitive ideals P_β are in one-to-one correspondence with a set of primitive ideals P'_β in E containing A, in such a way that $P_\beta = P'_\beta/A$ [4, p. 205]. We observe that $\bigcap_\beta P'_\beta = A$. Furthermore, the primitive ideals M_α are in one-to-one correspondence with a set of primitive ideals M'_α in E not containing A, in such a way that $M_\alpha = M'_\alpha \cap A$ [4, p. 206]. Thus

$$\left(\bigcap_\alpha M'_\alpha\right) \cap \left(\bigcap_\beta P'_\beta\right) = \left(\bigcap_\alpha M'_\alpha\right) \cap A = \bigcap_\alpha (M'_\alpha \cap A) = \bigcap_\alpha M_\alpha = (0),$$

so E is isomorphic to a subdirect sum of the quotient rings modulo the M'_α and P'_β.

Now each M'_α is a maximal ideal in E. For if I is an ideal in E containing M'_α, then $I \cap A = M_\alpha = M'_\alpha \cap A$, since $I \cap A$ is an ideal in A containing M_α, which is a maximal ideal in A. Thus $IA \subset M'_\alpha$. But $A \not\subset M'_\alpha$, and M'_α is a prime ideal, so that $I \subset M'_\alpha$, whence $I = M'_\alpha$. It follows that $(M'_\alpha, A) = E$ for each α, and we can write

$$E/M'_\alpha = (M'_\alpha, A)/M'_\alpha \cong A/(M'_\alpha \cap A) = A/M_\alpha \cong A_\alpha.$$

Also, for each β, we have the isomorphisms

$$E/P'_\beta \cong E/A\big/P'_\beta/A \cong S/P_\beta \cong B_\beta.$$

Therefore E is isomorphic to a subdirect sum of the collection of rings $\{A_\alpha\} \cup \{B_\beta\}$.

COROLLARY 1. *Let* F *be a field, and let* A *and* S *be rings of* F*-valued functions on some domains* X *and* Y, *respectively, such that, for each* $x \in X$ ($y \in Y$) *and each* $v \in F$, *there is an* $a \in A$ ($s \in S$) *with* $a(x) = v$ ($s(y) = v$). *Then any extension* E *of* A *by* S *is a ring of* F*-valued functions on* $X \cup Y$ *such that, for each* $z \in X \cup Y$ *and each* $v \in F$, *there is an* $e \in E$ *with* $e(z) = v$.

Proof. A and S are subdirect sums of fields, which are primitive simple rings.

If a ring A has a property π definable in terms of ring operations, we write $A \ni \pi$.

Definition 2. A ring property π will be said to be (1) *divisible* if $A \ni \pi$ implies that $B \ni \pi$ for every quotient ring B of A, (2) *summable* if $A_\alpha \ni \pi$ for all A_α in some collection implies that $B \ni \pi$ for every subdirect sum B of the rings A_α.

Definition 3. A ring A will be called *strongly semisimple* if the intersection of the primitive maximal ideals in A is (0).

Of course, every strongly semisimple ring is semisimple; and, since a commutative primitive ring is simple, every commutative semisimple ring is strongly semisimple.

THEOREM 5. *Let* E *be any extension of a strongly semisimple ring* A *by a semisimple ring* S. *For any divisible and summable ring property* π, $A \ni \pi$ *and* $S \ni \pi$ *imply that* $E \ni \pi$.

Proof. Since A and S are semisimple, they are isomorphic to subdirect sums of collections of primitive rings $\{A_\alpha\}$ and $\{B_\beta\}$, respectively; and we may assume that the A_α are simple, since A is strongly semisimple. By Theorem 4, E is isomorphic to a subdirect sum of the collection $\{A_\alpha\} \cup \{B_\beta\}$. Because π is divisible, $A_\alpha \ni \pi$ for all α and $B_\beta \ni \pi$ for all β. The fact that π is summable then implies that $E \ni \pi$.

COROLLARY 2. *If* A *and* S *are Boolean rings* (p-*rings*), *and* E *is any extension of* A *by* S, *then* E *is a Boolean ring* (p-*ring*).

Proof. Boolean rings (p-rings) are commutative and semisimple, and the property of being a Boolean ring (p-ring) is divisible and summable.

COROLLARY 3. *Let* E *be any extension of a strongly semisimple ring* A *by a semisimple ring* S. *If* A *and* S *satisfy a certain polynomial identity, then so does* E.

Proof. The property of satisfying a certain polynomial identity is divisible and summable.

COROLLARY 4. *Let* E *be any extension of a strongly semisimple ring* A *by a semisimple ring* S. *If* A *satisfies the polynomial identity* $P(x_1, \cdots, x_n) = 0$ *and* S *satisfies the polynomial identity* $Q(y_1, \cdots, y_m) = 0$, *then* E *satisfies the polynomial identity* $P(x_1, \cdots, x_n)Q(y_1, \cdots, y_m) = 0$.

Proof. A and S also satisfy the identity $P(x_1, \cdots, x_n)Q(y_1, \cdots, y_m) = 0$, so Corollary 3 applies.

Remark 3. If E is any extension of A by S, the requirement that A and S be commutative is necessary for the commutativity of E. From Corollary 3, this condition is sufficient if A and S are semisimple. The following example shows that it is not sufficient if A is not semisimple. Let A be the zero-ring whose additive group is the additive group of the real field R, and let S = R. Let E be the extension of A by S with the following operations:

$$(a, s) + (b, t) = (a + b, s + t), \qquad (a, s)(b, t) = (ab + sb, st) = (sb, st),$$

where sb is defined to be the element of R obtained by forming the usual product sb in R. It is easy to verify directly that E is a ring. Obviously, A and S are commutative; but E is not commutative, since $(0, 1)(1, 0) = (1, 0)$ while $(1, 0)(0, 1) = (0, 0)$. (E can be defined alternatively to be the algebra generated over R by the elements e and z satisfying the relations $e^2 = e$, $z^2 = 0$, $ez = z$, and $ze = 0$. This is similar to [4, p. 42].)

Remark 4. If E is an extension of A by S, and E has finite characteristic p, then A and S have characteristic p. Corollary 3 shows that if A is strongly semisimple, S is semisimple, and both rings have characteristic p, then E has characteristic p. But this also fails if A is not strongly semisimple (even if both rings are commutative): I_4 is an extension of the ring of even integers modulo 4 by I_2, but it does not have characteristic 2.

We turn now to the property of possessing an identity. Our theorem is a strengthened version of a theorem of Szendrei [8], and the proof is obtained by modifying his only slightly.

THEOREM 6. *If an extension* E *of* A *by* S *has an identity, then* (1) S *has an identity* u, *and* (2) *there exists* $b \in A$ *such that* $ua = a - ba$ *for all* $a \in A$. *Conversely, for any extension* E *of* A *by* S, *if* (1) *and* (2) *hold, and* A *has at least one element that is not a zero divisor, then* E *has an identity* (*namely*, (b, u)).

Proof. Assume that E has an identity (b, u). Since $S \cong E/A$, it is clear that u is the identity of S. For all $(a, 0) \in E$, we have the relations

$$(b, u)(a, 0) = (ba + ua, 0) = (a, 0),$$

whence $ua = a - ba$.

Now assume that conditions (1) and (2) hold, and that A has at least one element c that is not a zero divisor. For any $(a, s) \in E$,

$$(b, u)(a, s) = (ba + ua + bs + \{u, s\}, us).$$

Set $d = ba + ua + bs + \{u, s\}$. By conditions (11) and (9) in [8],

$$(bs)c = b(sc) \quad \text{and} \quad \{u, s\}c = u(sc) - (us)c,$$

so $cdc = c(ba + ua)c + c(b(sc) + u(sc) - sc)$. Using the assumed condition (2), we obtain $cdc = cac$. Since c is not a zero divisor, $d = a$. Thus,

$$(b, u)(a, s) = (d, us) = (a, s).$$

The proof that (b, u) is a right identity is similar.

Remark 5. Since any ring has an extension with an identity, no condition on A alone can be necessary. On the other hand, (1) and (2) by themselves are not sufficient: The example given in Remark 3 is an extension of a ring consisting of zero divisors by a ring with identity, and $ua = a = a - ba$ for any $b \in A$. But E does not have an identity, since $(1, 0)(a, s) = (0, 0)$ for all $(a, s) \in E$.

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