A RADICAL FOR NEAR-RING MODULES

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The theory of various radicals for near-rings has been discussed by Betsch [1], Deskins [3], and Laxton [7]. It is our purpose here to study a radical for near-ring modules which, when restricted to near-rings with identity, coincides with the radicals defined by Betsch and Laxton.

In the second section we show that if $J(M)$ is the radical of a module M over a near-ring R, then $J(M/J(M)) = 0$. Also, if A is a submodule of M and $J(M/A) = 0$, then $J(M) \subseteq A$. These results were first obtained by Betsch and Laxton in the special case of a near-ring with identity.

In the third section we introduce the concepts of small and strictly small submodules. If the radical $J(M)$ of a near-ring module M is small (or strictly small), then $J(M)$ is the intersection of all maximal submodules (or maximal R-subgroups). Furthermore, $J(M)$ is the sum of all small submodules of M if and only if every submodule of M generated by a finite subset of $J(M)$ is small.

In the fourth section we restrict our attention to near-ring modules M that satisfy the descending chain condition on submodules. If the radical is the zero submodule, then M is a finite direct sum of minimal submodules. Let M be a finitely generated R-module. The radical $J(M)$ of M is small if and only if every maximal submodule of M is maximal as an R-subgroup.

1. FUNDAMENTAL DEFINITIONS

Definition 1. A *near-ring* R is a system with two binary compositions, addition and multiplication, such that

(i) the elements of R form a group R^+ under addition,

(ii) the elements form a semigroup under multiplication,

(iii) $x(y + z) = xy + xz$, for all $x, y, z \in R$,

(iv) $0 \cdot x = 0$, where 0 is the additive identity of R^+ and x is an element of R.

In particular, if R contains a multiplicative semigroup S whose elements generate R^+ and satisfy the condition

(v) $(x + y)s = xs + ys$ for all $x, y \in R$ and $s \in S$,

then R is called a *distributively generated* (d. g.) near-ring.

The most natural example of near-rings is given by the set of identity-preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added by adding images, and multiplication is iteration, then the system $(R, +, \cdot)$ is a near-ring. The near-ring R is called the near-ring associated with G.

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Definition 2. A *near-ring module* M is a system consisting of an additive group M, a near-ring R, and a mapping $f: (m, r) \to mr$ of $M \times R$ into M such that

(i) $m(r + s) = mr + ms$, for all $m \in M$ and all $r, s \in R$,

(ii) $m(rs) = (mr)s$, for all $m \in M$ and all $r, s \in R$.

In addition, if R is a d.g. near-ring whose additive group R^+ is generated by a multiplicative semigroup S, then we shall assume

(iii) $(m_1 + m_2)s = m_1 s + m_2 s$, for all $m_1, m_2 \in M$ and $s \in S$.

Where no confusion can arise, we shall refer to a near-ring module M simply as an R-module.

Let R be the near-ring of mappings associated with an additive group G. Then G can be considered as an R-module.

An R-*homomorphism* is a mapping f of an R-module M into an R-module M' such that $(m + h)f = mf + hf$ and $(mf)r = (mr)f$, where $m, h \in M$ and $r \in R$. The *submodules* of an R-module M are defined to be kernels of R-homomorphisms of M. In particular, a submodule of the R-module R^+ is called a *right ideal* of the near-ring R.

The kernel K of an R-homomorphism f of an R-module M is an additive normal subgroup of M. Also,

$$[(m + k)r - mr]f = (mf + kf)r - (mf)r = 0 \in M',$$

for all $m \in M$, $k \in K$, and $r \in R$.

Suppose now B is any additive normal subgroup of an R-module M such that $(m + b)r - mr \in B$, for all $m \in M$, $b \in B$, and $r \in R$. If f is the natural group homomorphism of M onto M/B, then a simple calculation shows that the definition $(m + B)r = mr + B$ makes M/B into an R-module and that f can be considered as an R-homomorphism.

A subgroup H of an R-module M is said to be an R-*subgroup* if $HR \subset H$. A submodule B of an R-module M is called *strictly maximal* if B is maximal as an R-subgroup. A *minimal* R-module M is a nonzero R-module containing no proper nonzero R-subgroups. An *irreducible* R-module M is a nonzero R-module containing no nonzero submodules. Every minimal R-module is irreducible. However, there exist irreducible R-modules that are not minimal.

Let A be a nonempty subset of an R-module M. By the submodule (R-subgroup) *generated* by A we mean the intersection of all submodules (all R-subgroups) containing A. An R-module M is said to be *finitely generated (finitely generated as an* R-*subgroup of itself)* if it contains a finite subset A such that M is the submodule (the R-subgroup) generated by A.

The additive subgroup of an R-module M generated by a collection of submodules is a submodule. However, this is not true in general for R-subgroups (see [2]). If H is an R-subgroup and B is a submodule of an R-module M, then

$$H + B = \{h + b \mid h \in H,\ b \in B\}$$

is an R-subgroup.

An R-module M is said to be a *direct sum of submodules* if it is a direct sum of the corresponding additive normal subgroups.

2. DEFINITION AND PROPERTIES OF THE RADICAL

Definition 3. Let M be an R-module, and let I denote the set of all strictly maximal submodules of M. We define the submodule $J(M) = \bigcap_{B \in I} B$ to be the *radical* of M. It is understood that if I is empty, then M is its own radical, in which case we say M is a *radical module.*

THEOREM 1. *Let* M *be an* R*-module that is a direct sum of minimal submodules. Then* M *is not a radical module, and* $J(M) = 0$.

Proof. Let $M = \bigoplus_{\lambda \in \Omega} M_\lambda$, where M_λ is a minimal submodule and Ω an index set. If for each $\lambda_0 \in \Omega$ we define

$$B_{\lambda_0} = \bigoplus_{\lambda \in \Omega, \lambda \neq \lambda_0} M_\lambda,$$

then the relation $\bigcap_{\lambda \in \Omega} B_\lambda = 0$ is trivial. From the analogue of the second isomorphism theorem for operator groups [5, p. 136] it follows that M/B_λ is R-isomorphic to M_λ. Hence, B_λ is a strictly maximal submodule. This shows that M is not a radical module. Since $J(M) \subset \bigcap_{\lambda \in \Omega} B_\lambda$, it follows that $J(M) = 0$.

THEOREM 2. *If* M *is an* R*-module, then* $J(M/J(M)) = 0$.

Proof. If M is a radical module, the proof is trivial. Assume that J(M) is a proper submodule of M, and let f denote the natural R-homomorphism of M onto M/J(M). Then $J(M/J(M)) = \bigcap_{B \in I} Bf$. If $\bar{x} \in J(M/J(M))$ and B is a strictly maximal submodule of M, then $\bar{x} \in Bf$. Let x be a representative of $\bar{x}$. Then $x \in B$, and so $x \in J(M)$. Therefore, $\bar{x} = 0$ and $J(M/J(M)) = 0$.

THEOREM 3. *If* A *is a proper submodule of the* R*-module* M *and* $J(M/A) = 0$, *then* $J(M) \subset A$.

Proof. Let f denote the natural R-homomorphism of M onto M/A. Since

$$0 = J(M/A) = \bigcap_{B \supset A,\, B \in I} Bf,$$

we have the relation $A \supset J(M) = \bigcap_{B \in I} B$.

3. SMALL SUBMODULES

Definition 4. A submodule A of an R-module M is called *small (strictly small)* if $M = B$ for each other submodule (R-subgroup) B such that $M = A + B$.

Since every submodule of M is an R-subgroup, we have the following propositions.

LEMMA 1. *If* A *is strictly small, then* A *is small.*

LEMMA 2. *If* A *and* A' *are small (strictly small) submodules, then* $A + A'$ *is small (strictly small).*

Let L (L') denote the collection of maximal submodules (maximal R-subgroups) of the R-module M.

THEOREM 4. *If the radical* J(M) *is small (strictly small), then*

$$J(M) = \bigcap_{B \in L} B \; (J(M) = \bigcap_{B \in L'} B).$$

Proof. Assume J(M) is small, and let $A = \bigcap_{B \in L} B$. If $J(M) \not\subset A$, then there exists a maximal submodule B such that $J(M) \not\subset B$. From this it follows that $M = J(M) + B$, and therefore $M = B$, a contradiction.

Similarly, if J(M) is strictly small, then $J(M) = \bigcap_{B \in L'} B$.
From Lemma 1 and Theorem 4 we have the following proposition.

COROLLARY 1. *If* J(M) *is strictly small, then* $J(M) = \bigcap_{B \in L} B = \bigcap_{B \in L'} B$.

THEOREM 5. *Let* M *be a finitely generated* R-*module. The radical* J(M) *is small if and only if* $J(M) = \bigcap_{B \in L} B$.

Proof. Let $A = \bigcap_{B \in L} B$. If J(M) is small, then $J(M) = A$ by Theorem 4.

Assume that $J(M) = A$. If J(M) is not small, then there exists a proper submodule C such that $M = J(M) + C$. By Zorn's lemma, since M is a finitely generated R-module, there exists a maximal submodule B such that $C \subset B$. From this it follows that $M = J(M) + B$, and therefore $M = B$, a contradiction. Hence, J(M) is a small submodule.

By a similar argument, we can prove the following.

THEOREM 6. *Let* M *be an* R-*module that is finitely generated as an* R-*subgroup of itself. The radical* J(M) *is strictly small if and only if* $J(M) = \bigcap_{B \in L'} B$.

LEMMA 3. *If* A *is a small submodule of the* R-*module* M, *then* $A \subset J(M)$.

Proof. Assume A is a small submodule. If $A \not\subset J(M)$, then there exists a strictly maximal submodule B such that $A \not\subset B$, whence $M = A + B$. Since A is small, it follows that $M = B$, a contradiction.

From Lemma 3 we have the following.

COROLLARY 2. *The radical* J(M) *of an* R-*module* M *contains the sum of all small submodules of* M.

COROLLARY 3. *If the radical* J(M) *of an* R-*module* M *is small, then* J(M) *is the sum of all small submodules of* M.

Proof. Assume that J(M) is small. If A is the sum of all small submodules, then by Corollary 2, $A \subset J(M) \subset A$, and therefore $J(M) = A$.

COROLLARY 4. *If the radical* J(M) *is a strictly small submodule of* M, *then* J(M) *is the sum of all strictly small submodules.*

Proof. Assume that J(M) is strictly small. Let A denote the sum of all small submodules, and A' the sum of all strictly small submodules. By Lemma 1 and Corollary 3, $J(M) \subset A' \subset A \subset J(M)$, and therefore $J(M) = A'$.

THEOREM 7. *The radical* J(M) *of an* R-*module* M *is the sum of all small submodules if and only if every submodule* B *of* M *generated by a finite subset of* J(M) *is small.*

Proof. Let A denote the sum of all small submodules of M. If every submodule B generated by a finite subset of J(M) is small, then $J(M) \subset A$. By Corollary 2, we conclude that $J(M) = A$.

Conversely, let $J(M) = A$, let $x_1, \cdots, x_k$ be a finite subset of J(M), and let B be the submodule of M that is generated by the given set. Since $J(M) = A$, there exist small submodules $B_1, \cdots, B_n$ such that $B \subset \sum_{1 \leq i \leq n} B_i$. If B is not small, then there exists a proper submodule C such that $M = B + C$. From this it is evident that $M = \left(\sum_{1 \leq i \leq n} B_i\right) + C$. By Lemma 2, $M = C$, a contradiction.

By an analogous proof we obtain the following result.

THEOREM 8. *The radical* J(M) *of an* R-*module* M *is the sum of all strictly small submodules if and only if every submodule* B *generated by a finite subset of* J(M) *is strictly small.*

4. R-MODULES THAT SATISFY THE DESCENDING CHAIN CONDITION ON SUBMODULES

Throughout this section, M will denote an R-module that satisfies the descending chain condition on submodules.

THEOREM 9. *If the radical* $J(M) = 0$, *then* M *is expressible as a finite direct sum of minimal submodules.*

The proof is straightforward, and we omit it.

THEOREM 10. *If* $J(M) = 0$ *and* A *is a submodule of* M, *then there exists a submodule* B *such that* $M = A \oplus B$.

Proof. Assume $J(M) = 0$. By Theorem 9, $M = M_1 \oplus \cdots \oplus M_n$, where M_i is a minimal submodule of M. Let A be a submodule of M, and let Φ be the collection of all submodules C such that $A \cap C = 0$. By Zorn's lemma, Φ contains a maximal element B. Since M_i is minimal, it follows that either

$$M_i \cap (A \oplus B) = M_i \quad \text{or} \quad M_i \cap (A \oplus B) = 0.$$

However, if $M_i \cap (A \oplus B) = 0$, then $B + M_i$ is a submodule of M that contains B, and $A \cap (B + M_i) = 0$. This is a contradiction. Hence, $M_i \subset A \oplus B$, and therefore $M = A \oplus B$.

THEOREM 11. *If* $J(M) = 0$, *then every irreducible submodule of* M *is minimal.*

Proof. Assume that $J(M) = 0$, and let A be an irreducible submodule. By Theorem 9, $M = M_1 \oplus \cdots \oplus M_n$, where M_i is a minimal submodule. Let a be a nonzero element of A. If $a = m_1 + \cdots + m_n$, where $m_i \in M_i$, then there exists at least one index j such that $m_j \neq 0$. If f_j is the mapping that carries elements of M onto their components in M_j, then f_j is an R-homomorphism of M onto M_j. Since A is irreducible and M_j is minimal, f_j induces an R-isomorphism f of A onto M_j. From this it follows that A is a minimal submodule.

THEOREM 12. *If* $J(M) = 0$, *then every maximal submodule of* M *is strictly maximal.*

Proof. This follows immediately from Theorems 10 and 11.

THEOREM 13. *If* B *is a maximal submodule of* M *that contains the radical* J(M), *then* B *is strictly maximal.*

Proof. Let B be a maximal submodule of M that contains the radical J(M). If f is the natural R-homomorphism of M onto M/J(M), then Bf is a maximal submodule of the R-module M/J(M). By Theorems 2 and 12, Bf is strictly maximal. It now follows that B is a strictly maximal submodule of M.

THEOREM 14. *Let* M *be a finitely generated* R-*module. The radical* J(M) *is small if and only if every maximal submodule of* M *is strictly maximal.*

Proof. Assume J(M) is small, and let B be a maximal submodule. By Theorem 4, $J(M) \subset B$, and therefore B is strictly maximal because of Theorem 13.

The converse follows from Theorem 5, since M is finitely generated.

THEOREM 15. *Let* R *be a* d.g. *near-ring, and let* M *be a finitely generated* R-*module whose additive group* M^+ *is solvable. Then the radical* J(M) *of* M *is a small submodule.*

Proof. Let B be a maximal submodule of M. Suppose that the additive group $(M/B)^+$ of M/B is not abelian. Since $(M/B)^+$ is solvable, there exists an additive normal subgroup C of M^+ such that $B \subset C \subset M$, $B \neq C \neq M$, and C/B is the commutator subgroup of $(M/B)^+$. Now let S be the multiplicative semigroup generating R^+. Since the commutator subgroup is fully invariant, $C/B \cdot S \subset C/B$, hence $C \cdot S \subset C$. From this it follows that C is a submodule of M. But this contradicts the maximality of B. Therefore the additive group $(M/B)^+$ of M/B is abelian, and since M/B is an irreducible R-module, it is minimal. Hence B is strictly maximal. By Theorem 14, J(M) is small. (The proof of Theorem 15 is essentially due to the referee.)

THEOREM 16. *Let* R *be a* d.g. *near-ring, and let* M *be an* R-*module whose additive group* M^+ *is finite and nilpotent. Then the radical* J(M) *is strictly small.*

Proof. Let B be a maximal R-subgroup of M. It is well known [6, p. 215] that B is a term of a normal series for the additive group M^+. From this it follows that there exists a proper additive normal subgroup C of M^+ such that $B \subset C$. Let C' denote the additive normal subgroup of M^+ generated by B. Now C' is a proper additive subgroup, and the elements c' of C' are finite sums of the form $\sum_i (m_i + b_i - m_i)$, where $m_i \in M$, $b_i \in B$, for all i. If S denotes the multiplicative semigroup that generates R^+, then for all $s \in S$ we have the relation $\left(\sum_i (m_i + b_i - m_i)\right) s = \sum_i (m_i s + b_i s - m_i s)$. Since the right-hand member is contained in C', it follows that C' is a submodule of the R-module M, and therefore $B = C'$. Since M is a finite group, by Theorem 6 J(M) is strictly small.

5. REMARKS AND EXAMPLES

Definition 5. A nonempty subset B of a near-ring R is said to be *nilpotent* if there exists a positive integer n such that $b_1 \cdots b_n = 0$ for all sequences $[b_1, \cdots, b_n]$ of elements from B [4, p. 88].

Let R be a finite d.g. near-ring with identity. R. R. Laxton [7, Theorem 3.5, p. 48] proved that the radical J(R) of R is nilpotent if and only if every maximal right ideal of R is strictly maximal. By Theorem 14, J(R) is nilpotent if and only if it is small. By Theorem 15, we obtain the following example of Laxton [7, p. 49].

EXAMPLE 1. *If* R *is a finite* d.g. *near-ring with identity whose additive group* R^+ *is solvable, then the radical* J(R) *is nilpotent and small.*

Let G be a finite, nonabelian, simple group, and R the d.g. near-ring generated by the inner automorphisms of G. For each $x \in R$, let f_x be the mapping of R^+ defined by $yf_x = xy$ for all $y \in R$. By the left distributive law, f_x is an endomorphism of R^+. We now give an example due to Laxton [8, p. 16].

EXAMPLE 2. *Let* T *denote the near-ring generated by* $\{f_x \mid x \in R^+\}$. *Then the radical* J(T) *of the distributively generated near-ring* T *is nonnilpotent. Hence,* J(T) *is not small.*

Because of Theorem 14, the near-ring T contains a maximal right ideal B that is not strictly maximal. Therefore, we have the following.

EXAMPLE 3. *The* T-*module* T/B *is irreducible but not minimal.*

Let G be a finite, nonabelian p-group, and R the d.g. near-ring generated by the multiplicative semigroup of endomorphisms of G. Then the additive order of the identity endomorphism is a power of p, and therefore R^+ is a finite, nonabelian p-group. It is well known [6, p. 216] that the group R^+ is nilpotent. By Theorem 16, we obtain the next example.

EXAMPLE 4. *The radical of* R *is strictly small.*

REFERENCES

1. G. Betsch, *Ein Radikal für Fastringe*, Math. Z. 78 (1962), 86-90.

2. D. W. Blackett, *Simple and semisimple near-rings*, Proc. Amer. Math. Soc. 4 (1953), 772-785.

3. W. E. Deskins, *A radical for near-rings*, Proc. Amer. Math. Soc. 5 (1954), 825-827.

4. A. Fröhlich, *Distributively generated near-rings* (I. *Ideal theory*), Proc. London Math. Soc. (3) 8 (1958), 76-94.

5. N. Jacobson, *Lectures in abstract algebra*, Vol. I., *Basic concepts*, Van Nostrand, New York, 1951.

6. A. G. Kurosh, *The theory of groups*, Vol. II, Chelsea, New York, 1956.

7. R. R. Laxton, *A radical and its theory for distributively generated near-rings*, J. London Math. Soc. 38 (1963), 40-49.

8. ———, *Prime ideals and the ideal radical of a distributively generated near-ring*, Math. Z. 83 (1964), 8-17.

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