A GAP-THEOREM FOR ENTIRE FUNCTIONS OF INFINITE ORDER

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1. INTRODUCTION AND NOTATION

Let $f(z) = \Sigma a_n z^{\lambda_n}$ be an entire function, and write

$$M(r, f) = \max_{|z|=r} |f(z)|, \qquad m(r, f) = \min_{|z|=r} |f(z)| .$$

In a recent paper, W. H. J. Fuchs [2] proved that if $f(z)$ is of finite order and the sequence $\{\lambda_n\}$ satisfies the "Fabry" gap condition

$$\frac{\lambda_n}{n} \to \infty, \tag{1}$$

then, for each $\varepsilon > 0$, the inequality

$$\log m(r, f) > (1 - \varepsilon) \log M(r, f) \tag{2}$$

holds outside a set of logarithmic density 0.

For functions of infinite order, (1) certainly does not imply (2). In fact, for every sequence $\{\lambda_n\}$ satisfying the condition

$$\sum_1^\infty \frac{1}{\lambda_n} = \infty,$$

A. J. Macintyre [5] has constructed an entire function bounded on the positive real axis. In this paper I shall prove that if the gap condition (1) is replaced by the more stringent condition

$$\lambda_n > n(\log n)^{2+\eta} \tag{3}$$

(for some $\eta > 0$), then (2) holds also for functions of infinite order. It would be desirable to replace condition (3) by the "exact" condition

$$\sum_1^\infty \frac{1}{\lambda_n} < \infty;$$

but this is beyond the scope of our method. The most that could possibly be squeezed out of our method is the replacement of (3) by the condition

$$\lambda_n > n(\log n)(\log \log n)^{2+\eta}.$$

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THEOREM. *If the exponents of* f(z) *satisfy the gap-condition* (3), *then* (2) *holds outside a set of finite logarithmic measure.*

The proof is similar to that of [2].

We shall use the notation

$$m^*(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})|\, d\phi,$$

$n(r, 0), n(r, \infty)$ = number of zeros (respectively, poles) in $|z| \leq r$,

$$N(r, 0) = \int_0^r \frac{n(t, 0)}{t}\, dt,$$

$$T(r, f) = m(r, f) + N(r, \infty),$$

$$M(r, f, \theta, \delta) = \max_{|\phi - \theta| \leq \delta/2} |f(re^{i\phi})|.$$

We shall assume throughout that $f(0) = 1$.

2. AUXILIARY PROPOSITIONS

LEMMA 1 [7, p. 30]. *If* $\{\lambda_n\}$ *is a strictly increasing sequence of nonnegative integers, then for all* θ *and* δ $(0 \leq \theta < 2\pi,\ 0 < \delta \leq 2\pi)$,

$$\max_{0 \leq \phi < 2\pi} \left| \sum_{n=1}^{M} A_n e^{i\lambda_n \phi} \right| \leq \left(\frac{40}{\delta}\right)^M \max_{|\phi - \theta| \leq \delta/2} \left| \sum_{n=1}^{M} A_n e^{i\lambda_n \phi} \right|.$$

The following lemma is a special case of [1, Lemma 10.1].

LEMMA 2. *Let* S(x) *be an increasing, continuous, positive function of* x *(for* $0 < x < \infty$*), and let* $\mu(y)$ *be an increasing, continuous, positive function of* y *(for* $0 < y < \infty$*), such that*

$$\int_1^\infty \frac{dy}{\mu(y)} < \infty; \tag{4}$$

then the set

$$E = \left\{ x \mid S\left(x + \frac{1}{\mu(S(x))}\right) > S(x) + h \right\}$$

is of finite measure, for every $h > 0$.

Proof. For a fixed positive value of h, let x_0 denote the least value of x satisfying the inequality

$$S\left(x + \frac{1}{\mu(S(x))}\right) \geq S(x) + h, \tag{5}$$

and write $\xi_0 = x_0 + 1/\mu(S(x_0))$. After $x_0, \cdots, x_{n-1}$ and $\xi_0, \cdots, \xi_{n-1}$ have been defined, let x_n be the least value x in $[\xi_{n-1}, \infty)$ that satisfies (5), and let $\xi_n = x_n + 1/\mu(S(x_n))$. Then clearly

$$0 \leq x_0 < \xi_0 \leq x_1 < \xi_1 \leq \cdots,$$

$$S(x_n) \geq S(\xi_{n-1}) > S(x_{n-1}) + h \geq \cdots \geq S(x_0) + nh \geq nh, \tag{6}$$

$$\xi_n - x_n = \frac{1}{\mu(S(x_n))} \leq \frac{1}{\mu(nh)},$$

$$\sum_{n=2}^{\infty} (\xi_n - x_n) \leq \sum_{n=2}^{\infty} \frac{1}{\mu(nh)} < \frac{1}{h}\int_h^{\infty} \frac{dy}{\mu(y)} < \infty. \tag{7}$$

In view of (6), clearly $x_n \to \infty$, and therefore E is covered by the intervals (x_n, ξ_n). The result now follows from (7).

LEMMA 3. *If* $Q(r)$ *is an increasing positive function for* $r > 1$, *then for every* $\varepsilon > 0$ *and* $q > 1$,

$$Q\left(r + \frac{r}{\log^{1+\varepsilon} Q(r)}\right) < q\,Q(r)$$

outside an exceptional set of finite logarithmic measure.

Proof. Writing

$$S(x) = \log Q(e^x), \qquad \mu(y) = y^{1+\varepsilon}, \qquad h = \log q,$$

and using the inequality $1 + u \leq e^u$, we obtain this lemma immediately from Lemma 2.

LEMMA 4. *Let* $f(z) = \Sigma\, a_n z^n$ *be an entire function, let* $r = |z|$ *and* $\omega > 0$, *and let* ν *and* $R(z)$ *be defined by the equations*

$$\nu = [3 \log M(r) \cdot (\log\log M(r))^{1+\omega}], \tag{8}$$

$$R(z) = \sum_{n=\nu+1}^{\infty} a_n z^n.$$

Then, outside an exceptional set of finite logarithmic measure, $|R(z)| \leq 1$.

Proof. With $r < \rho$, we have the inequalities

$$|a_n| \leq \frac{M(\rho)}{\rho^n},$$

$$|R(z)| \leq M(\rho) \sum_{n=\nu+1}^{\infty} \left(\frac{r}{\rho}\right)^r = M(\rho)\left(\frac{r}{\rho}\right)^{\nu+1} \frac{\rho}{\rho - r},$$

$$\begin{aligned}\log|R(z)| &\leq \log M(\rho)+(\nu+1)\log\left(1-\frac{\rho-r}{\rho}\right)+\log\frac{\rho}{\rho-r}\\ &\leq \log M(\rho)-(\nu+1)\frac{\rho-r}{\rho}+\log\frac{\rho}{\rho-r}.\end{aligned} \tag{9}$$

Putting $\rho = r\left\{1+\frac{1}{(\log\log M(r))^{1+\omega}}\right\}$, we obtain from (8) and (9) the inequality

$$\log|R(z)| \leq \log M(\rho) - 3\log M(r)\cdot\frac{r}{\rho} + (1+\omega)\log\log\log M(r) + \log\frac{\rho}{r}.$$

Applying Lemma 3 to the function $Q(r) = \log M(r)$, with $q = e$, we get the bound

$$\log M(\rho) = \log M\left(r+\frac{r}{(\log\log M(r))^{1+\omega}}\right) < e\cdot\log M(r)$$

outside a set E of finite logarithmic measure. Hence, for $r \notin E$ and $r > r_0$,

$$\log|R(z)| \leq -\frac{1}{10}\log M(r) + (1+\omega)\log\log\log M(r) + 1 < 0.$$

The following Lemma is an adaptation of [4, Lemma VIII].

LEMMA 5. *Let* $f(z) = \Sigma a_n z^{\lambda_n}$ *be an entire function satisfying the gap-condition* (3). *Let* θ_r *and* δ_r *be functions of* r, *subject only to the condition that*

$$\delta_r \geq (\log M(r))^{-\gamma} \tag{10}$$

for some $\gamma > 0$. *Then, for every* $\varepsilon > 0$,

$$\log M(r, \theta_r, \delta_r) > (1-\varepsilon)\log M(r) \tag{11}$$

outside an exceptional set of finite logarithmic measure.

Proof. Clearly, (3) implies that

$$n \leq 2\lambda_n(\log\lambda_n)^{-2-\eta}.$$

Put $\omega = \eta/2$, and define ν by (8). If $\lambda_\ell \leq \nu < \lambda_{\ell+1}$, then

$$\ell \leq 2\lambda_\ell(\log\lambda_\ell)^{-2-\eta} < 2\nu(\log\nu)^{-2-\eta}$$

$$< 6\log M(r)\cdot(\log\log M(r))^{1+\frac{1}{2}\eta}\cdot\left\{\log 3+\log\log M(r)+\left(1+\tfrac{1}{2}\eta\right)\log\log\log M(r)\right\}^{-2-\eta}$$

$$< 6\log M(r)\cdot(\log\log M(r))^{1+\frac{1}{2}\eta}\cdot\left\{\frac{4}{5}\log\log M(r)\right\}^{-2-\eta}$$

$$< 12\log M(r)\cdot(\log\log M(r))^{-1-\frac{1}{2}\eta} \qquad \text{for } \eta < 1,\ r > r_0.$$

We now apply Lemma 1, and using condition (10), we obtain (for $x > x_0$) the inequalities

$$\max_{\phi}\left|\sum_{\lambda_n\le\nu} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right| = \max_{\phi}\left|\sum_{n=1}^{\ell} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right|$$

$$\le \exp\left\{\ell\cdot\log\frac{40}{\delta_r}\right\}\cdot \max_{|\phi-\theta|\le\delta/2}\left|\sum_{n=1}^{\ell} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right|$$

$$\le \exp\left\{12\log M(r)\cdot(\log\log M(r))^{-1-\frac{1}{2}\eta}\cdot(\log 40+\gamma\log\log M(r))\right\}$$

$$\max_{|\phi-\theta|\le\delta/2}\left|\sum_{\lambda_n\le\nu} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right|$$

$$\le \exp\left\{A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}\right\}\cdot \max_{|\phi-\theta|\le\delta/2}\left|\sum_{\lambda_n\le\nu} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right|.$$

Combining this result with Lemma 4, we see that

$$M(r) \le \max_{|z|=r}\left|\sum_{\lambda_n\le\nu} a_n z^{\lambda_n}\right| + \max_{|z|=r}\left|\sum_{\lambda_n\ge\nu+1} a_n z^{\lambda_n}\right|$$

$$\le \exp\left\{A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}\right\}\cdot \max_{|\phi-\theta|\le\delta/2}\left|\sum_{\lambda_n\le\nu} a_n r^{\lambda_n} e^{i\lambda_n\phi}\right| + 1$$

$$\le \exp\left\{A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}\right\}\cdot\{M(r,\ \theta,\ \delta)+1\}+1$$

$$\le \exp\left\{A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}\right\}\cdot\{M(r,\ \theta,\ \delta)+2\}$$

$$\le \exp\left\{A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}\right\}\cdot M(r,\ \theta,\ \delta)+o(M(r))\cdot\log M(r)+o(1)$$

$$\le A\cdot\log M(r)\cdot(\log\log M(r))^{-\frac{1}{2}\eta}+\log M(r,\ \theta,\ \delta) = o(\log M(r))+\log M(r,\ \theta,\ \delta)$$

outside an exceptional set of finite logarithmic measure. Thus we have shown that

$$\log M(r,\ \theta,\ \delta) = \{1+o(1)\}\cdot\log M(r)$$

outside an exceptional set of finite logarithmic measure, and this proves Lemma 5.

The following is an adaptation of [1, Lemma 2].

LEMMA 6. *Let* f(z) *be a meromorphic function (of finite or infinite order), and let* $\{a_k\}$ *and* $\{b_k\}$ *be the sequences of its zeros and poles, respectively, each zero or pole appearing as often as its multiplicity indicates. Let* $\{d_m\}$, *the combined sequence of zeros and poles, be arranged according to increasing modulus. Let* Γ *be the countable union of the (eccentric) discs*

$$\left|\frac{z - d_m}{z}\right| < \frac{1}{2m^2}.$$

Then, if $z \notin \Gamma$ *and* $r_0 < |z| < R < 2|z|$,

$$\left|\frac{zf'(z)}{f(z)}\right| < A\left(\frac{R}{R - |z|}\right)^3 T(R, f)^3 . \tag{12}$$

Proof. The Jensen-Nevanlinna identity, with $\rho = \dfrac{R + |z|}{2}$, is the formula

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\phi})| \frac{2\rho e^{i\phi}}{(\rho e^{i\phi} - z)^2}\, d\phi$$

$$+ \sum_{|a_n|<\rho} \frac{\rho^2 - |a_n|^2}{(\rho - \bar{a}_n z)(z - a_n)} - \sum_{|b_k|<\rho} \frac{\rho^2 - |b_k|^2}{(\rho^2 - \bar{b}_k z)(z - b_k)}.$$

Using the inequalities

$$\frac{\rho^2 - |c|^2}{|\rho^2 - \bar{c}z|^2} \le \frac{\rho^2 - |c|^2}{\rho|z| - |c||z|} = \frac{\rho + |c|}{|z|} \le \frac{2\rho}{|z|} \qquad (\text{for } |c| < \rho),$$

and

$$\left|\frac{2\rho e^{i\phi}}{(\rho e^{i\phi} - z)^2}\right| < \frac{2\rho}{(\rho - |z|)^2},$$

we obtain the bound

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{2\rho|z|}{(\rho - |z|)^2}\left\{m(\rho, f) + m\left(\rho, \frac{1}{f}\right)\right\} + 2\rho \sum_{|d_m|\le\rho} \frac{1}{|z - d_m|}.$$

If $z \notin \Gamma$, we deduce that (with $|z| = r$)

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{2\rho r}{(\rho - r)^2}\left\{m(\rho, f) + m\left(\rho, \frac{1}{f}\right)\right\} + \frac{4\rho}{r} \sum_{|d_m|\le\rho} m^2.$$

Since

$$m(\rho, f) + m\left(\rho, \frac{1}{f}\right) \le 2T(\rho) + O(1) < 3T(\rho) < 3T(R)$$

(by Nevanlinna's first fundamental theorem), and since

$$\sum_{|d_m|<\rho} m^2 \le \{n(\rho, 0) + n(\rho, \infty)\}^3 = n^3(\rho),$$

$$\frac{R - x}{2R}\, n(\rho) = \frac{R - \rho}{R}\, n(\rho) \leq \int_\rho^R \frac{n(t)}{t}\, dt \leq N(R, 0) + N(R, \infty) \leq 2T(R) + O(1)$$

and

$$\frac{2\rho r}{(\rho - r)^2} \leq \frac{8R^2}{(R - r)^2} \qquad \left(\frac{4\rho}{r} \leq \frac{4R}{r}\right),$$

we see that

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 24\left(\frac{R}{R - r}\right)^2 T(R) + 256\,\frac{R}{r}\left(\frac{R}{R - r}\right)^3 T(R)^3 \leq 536\left(\frac{R}{R - r}\right)^3 T(R)^3.$$

LEMMA 7. *For any entire function* $f(z)$,

(13)
$$\left|\frac{zf'(z)}{f(z)}\right| < A \cdot \log^4 M(r, f) \qquad (r = |z|)$$

outside an exceptional set of finite logarithmic measure.

Proof. The disc $\left|\frac{z - d}{z}\right| < \frac{1}{2m^2}$ is contained in the annulus

$$d\left(1 + \frac{1}{2m^2}\right)^{-1} < |z| < d\left(1 - \frac{1}{2m^2}\right)^{-1}.$$

Hence Γ, the exceptional set of Lemma 6, is contained in the union E of the annuli

$$d_m\left(1 + \frac{1}{2m^2}\right)^{-1} < |z| < d_m\left(1 - \frac{1}{2m^2}\right)^{-1}.$$

If E^* is the intersection of E with the positive real axis, then the logarithmic measure of E^* is

$$\sum_{m=1}^{\infty} \log \frac{1 + \frac{1}{2m^2}}{1 - \frac{1}{2m^2}} < \sum_{m=1}^{\infty} \frac{2}{m^2} = \frac{\pi^2}{3}.$$

In (12) we now write $R = r + \frac{r}{\log^2 T(r)}$, $r = |z|$, and we apply Lemma 3 with $Q(r) = T(r)$, $q = e$, and $\varepsilon = 1$. If E_0 is the exceptional set of Lemma 3, then (12) and the inequality of Lemma 3 hold simultaneously for $r \notin E_0 \cup E^*$, and we deduce that

$$\left|\frac{zf'(z)}{f(z)}\right| < A\,(R/r)^3 \log^6 T(r) \cdot e^4 \cdot T(r)^3 < A\,T(r)^4 \qquad \text{for } r > r_0.$$

Since $T(r, f) \leq \log M(r, f)$, for entire functions, the lemma follows at once.

3. PROOF OF THE THEOREM

We can now easily prove our theorem. If we write

$$\delta_r = (\log M(r))^{-4},$$

(11) and (13) hold simultaneously outside an exceptional set E of finite logarithmic measure. For each ϕ there exists by Lemma 5 a real ψ such that

$$|\phi - \psi| < \delta_r = (\log M(r))^{-4} \quad \text{and} \quad \log|f(re^{i\psi})| > (1 - \varepsilon/2)\log M(r).$$

Now, using (13), we deduce that

$$\log|f(re^{i\phi})| = \log|f(re^{i\psi})| + \int_\psi^\phi \frac{\partial}{\partial\theta}\log|f(re^{i\theta})|\,d\theta$$

$$\geq (1 - \varepsilon/2)\log M(r) - \int_\psi^\phi r\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right||d\theta| > (1 - \varepsilon/2)\log M(r) - A\delta_r \log^4 M(r)$$

$$= (1 - \varepsilon/2)\log M(r) - A > (1 - \varepsilon)\log M(r)$$

for $r > r_0$, $r \notin E$. This proves the theorem.

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