SOME RINGS WITH NIL COMMUTATOR IDEALS

Thomas P. Kezlan

In a previous paper Drazin investigated certain conditions on a ring which would guarantee that its commutator ideal be nil [1]. In particular, he considered a class of rings, which he called K-rings, satisfying a certain Engel condition. However, as has been pointed out, the proof of one of the main theorems contains an error, and this in turn invalidates the proofs of several of the subsequent results [2], [3], [4]. In the present paper we shall prove one of these results and in addition obtain some related theorems. It should be mentioned that only a few of the proofs in [1] are invalid [2], [3]; indeed, in this paper we shall make use of several key results of [1], namely, Lemma 4.2 and Theorems 4.2 and 4.3.

We begin by recalling several classes of rings defined by Drazin in [1]. Let x and y be elements of a ring R. We define

$$e_0(x, y) = x, \qquad e_1(x, y) = [x, y] = xy - yx,$$

$$e_k(x, y) = [e_{k-1}(x, y), y] \qquad (k = 1, 2, \cdots).$$

Equivalently (proved easily by induction on k),

$$e_k(x, y) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^j x y^{k-j}.$$

If m is a positive integer, then a ring R is called an m-*ring* if and only if for every x, y in R, there exist integers k, t, n, and q and an element a in R such that $1 \leq t \leq m$ and

$$x^{m-t} e_k(qx^{t+1} + x^{t+1} a - x^t, y^n) = 0.$$

R is called a K-*ring* if and only if for every x, y in R, there exist integers $k = k(x, y)$ and $n = n(x, y)$ such that $e_k(x, y^n) = 0$. Thus every K-ring is a 1-ring, and if $t \leq m$, then every t-ring is an m-ring. Clearly, the properties of being a K-ring or an m-ring are preserved under homomorphism.

If R is any ring, the *Levitzki radical* of R, that is, the sum of all locally nilpotent ideals of R, will be denoted L(R) [9]. L(R) contains every locally nilpotent left (or right) ideal of R [5, p. 27]. The *Köthe radical* of R, that is, the sum of all nil ideals of R, will be denoted N(R) [8]. If J(R) denotes the Jacobson radical of R, then $L(R) \subset N(R) \subset J(R)$. It can easily be verified that the following assertions are equivalent for any ring R:

(i) the nilpotent elements of R form an ideal of R;

(ii) N(R) is precisely the set of nilpotent elements of R;

(iii) every nilpotent element of R generates a nil ideal of R;

(iv) R/N(R) has no nonzero nilpotent elements.

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We shall denote by $C(R)$ the *commutator ideal* of R, that is, the ideal of R generated by all commutators $[x, y]$, with x, y in R. The following can easily be proved:

(i) if $C(R)$ is a nil ideal, then the nilpotent elements of R form an ideal;

(ii) $C(R)$ is a nil ideal if and only if $R/N(R)$ is commutative.

We shall say that R is of *characteristic* 0 if and only if $mx \neq 0$ for each nonzero element x in R and each nonzero integer m. Lemmas 1, 2, and 3 below can be proved easily by induction on k. The proofs will be omitted.

LEMMA 1. *Let* R *be a ring, and let* k *and* n *be positive integers. Then there exist polynomials* $f_0, f_1, \cdots, f_{nk-1}$ *in two indeterminates over* R *such that for each integer* λ *and all* x, y *in* R,

$$e_k(y, (x+\lambda y)^n) = \sum_{i=0}^{nk-1} \lambda^i f_i(x, y),$$

where $f_{nk-1}(x, y) = -e_k(x, y^n)$.

LEMMA 2. *Let* k *be a nonnegative integer and* n *a positive integer. Define a sequence of integers* $A^j_{n,k}$ $(j = 0, \pm 1, \pm 2, \cdots)$ *as follows:*

$$A^j_{n,k} = \begin{cases} 0 & \text{if } j < 0 \text{ or } j > k(n-1), \\ 1 & \text{if } j = 0, \\ A^j_{n,k-1} + A^{j-1}_{n,k-1} + \cdots + A^{j-(n-1)}_{n,k-1} & \textit{otherwise}. \end{cases}$$

If R *is any ring and* x, y *are elements of* R, *then*

$$e_k(x, y^n) = \sum_{j=0}^{k(n-1)} A^j_{n,k} y^j e_k(x, y) y^{k(n-1)-j}.$$

In particular, if $e_k(x, y) = 0$, *then* $e_k(x, y^n) = 0$.

LEMMA 3. *If* k *is a nonnegative integer, then for any* x, y, *and* z *in a ring* R,

$$e_k(xy, z) = \sum_{j=0}^{k} \binom{k}{j} e_j(x, z) e_{k-j}(y, z).$$

If R is a K-ring and x, y are in R, we shall always assume that $k(x, y)$ is the smallest nonnegative integer for which there exists a positive integer n such that $e_{k(x,y)}(x, y^n) = 0$. We then define $n(x, y)$ as the smallest positive integer for which $e_{k(x,y)}(x, y^{n(x,y)}) = 0$.

THEOREM 1. *Let* R *be a* K*-ring whose nilpotent elements form an ideal and which has characteristic* 0. *Suppose* R *satisfies either*

(a) k *is independent of* x, *or*

(b) k *and* n *are independent of* y.

Then $C(R)$ *is a nil ideal.*

Proof. We may assume that $N(R) = (0)$; for suppose we have established the theorem in this case. Then in the general case it is easily seen that $R/N(R)$ has characteristic 0 and also inherits the other hypotheses of the theorem. Thus, since $N(R/N(R)) = (0)$, we see that $C(R/N(R))$ is a nil ideal of $R/N(R)$. But $R/N(R)$ has no nonzero nil ideals and hence is commutative, which shows that $C(R)$ is nil.

Assuming then that $N(R) = (0)$, in other words, that R has no nonzero nilpotent elements, we shall suppose that there exist x, y in R with $k(x, y) > 1$, and arrive at a contradiction. Let

$$k_1 = k(x, y), \qquad n_1 = n(x, y),$$

$$k_2 = k(x^2, y), \qquad n_2 = n(x^2, y).$$

Using Lemma 3, we see that

$$e_{2k_1-1}(x^2, y^{n_1}) = 0, \tag{1}$$

and from Lemmas 2 and 3 we obtain

$$e_{2k_1-2}(x^2, y^{n_1 n_2}) = \binom{2k_1 - 2}{k_1 - 1} (e_{k_1-1}(x, y^{n_1 n_2}))^2. \tag{2}$$

From (1) we conclude that $k_2 \leq 2k_1 - 1$. Now, for $k_2 \leq 2k_1 - 2$, condition (2) implies that

$$\binom{2k_1 - 2}{k_1 - 1} (e_{k_1-1}(x, y^{n_1 n_2}))^2 = 0,$$

and since R has characteristic 0 and contains no nonzero nilpotent elements, it would follow that $e_{k_1-1}(x, y^{n_1 n_2}) = 0$, contrary to the minimality of $k_1 = k(x, y)$. Hence $k_2 = 2k_1 - 1$; that is, $k(x^2, y) = 2k(x, y) - 1$. In the same way we obtain the relation

$$k(x^{2^{m+1}}, y) = 2k(x^{2^m}, y) - 1 \qquad (m = 0, 1, \cdots).$$

In particular,

(3) the set of integers $k(x^{2^m}, y)$ $(m = 0, 1, \cdots)$ is unbounded.

If hypothesis (a) holds, there exists $k(y)$ such that $k(z, y) \leq k(y)$ for every z in R. But then $k(x^{2^m}, y) \leq k(y)$ for $m = 0, 1, \cdots$, contrary to (3). Thus if (a) holds, the assumption $k(x, y) > 1$ leads to a contradiction.

Now suppose (b) holds. There exist $k = k(y)$ and $n = n(y)$ such that $e_k(y, z^n) = 0$ for every z in R. Hence for any integer λ and z in R, it follows by Lemma 1 that

$$0 = e_k(y, (z + \lambda y)^n) = \sum_{i=0}^{nk-1} \lambda^i f_i(z, y),$$

where $f_{nk-1}(z, y) = -e_k(z, y^n)$. Taking successively $\lambda = 0, 1, \cdots, nk - 1$, we obtain a linear homogeneous system of nk equations in the nk "unknowns" $f_0, f_1, \cdots, f_{nk-1}$:

$$\sum_{i=0}^{nk-1} \lambda^i f_i(z, y) = 0 \qquad (\lambda = 0, 1, \cdots, nk - 1).$$

This system has a Vandermonde (and hence nonzero) determinant; therefore it has no nontrivial solution over R, since R has characteristic 0. In particular, $f_{nk-1}(z, y) = 0$; that is, $e_k(z, y^n) = 0$ for every z in R. But then $k(z, y) \leq k = k(y)$ for every z in R, and this again contradicts (3).

Therefore we conclude that $k(x, y) = 1$ for all x, y in R; that is, to each pair x, y in R there corresponds an integer $n = n(x, y)$ such that $xy^n = y^n x$. But it has been proved that C(R) is a nil ideal for such a ring R [1], [6]. This completes the proof of the theorem.

Let F(R) denote the ideal of R consisting of all elements of finite additive order. The following lemma appears in [1]; however, its proof is included here for reference.

LEMMA 4 (Amitsur). *Let* $\mathscr{R}$ *be a class of rings, and suppose that for each* R *in* $\mathscr{R}$ *there is given a subset* $Q(R) \subset R$. *Suppose further that*

(i) *for every* R *in* $\mathscr{R}$ *and every homomorphism* θ *of* R, $R\theta$ *belongs to* $\mathscr{R}$ *and* $Q(R)\theta \subset Q(R\theta)$;

(ii) Q(R) *is nil for all* R *in* $\mathscr{R}$ *whose characteristic is either* 0 *or a prime.*

Then Q(R) *is nil for every* R *in* $\mathscr{R}$.

Proof. Let R be in $\mathscr{R}$, and let c be in Q(R). We shall show that c is nilpotent. Since R/F(R) has characteristic 0 and is a homomorphic image of R, it follows from (ii) that Q(R/F(R)) is nil. By (i) we see that $Q(R)/F(R) \subset Q(R/F(R))$, whence some power c^i of c lies in F(R). Thus there exists a positive integer r such that $rc^i = 0$. Let s denote the smallest positive integer annihilating a power of c, say $sc^j = 0$. We shall show that $s = 1$. If $s \neq 1$, there exists a prime p dividing s, say $s = pt$, where $1 \leq t < s$. Since R/(pR) is in $\mathscr{R}$ and the characteristic of R/(pR) is p, it follows from (ii) that Q(R/(pR)) is nil. Thus $Q(R)/(pR) \subset Q(R/(pR))$ shows that some power of c lies in pR, say $c^k = pd$, where d is in R. But then

$$tc^{j+k} = ptc^j d = sc^j d = 0,$$

where $1 \leq t < s$, contrary to our choice of s. The result follows from this.

An inspection of the proof of Lemma 4 shows that hypothesis (i) need only be verified when θ is the natural homomorphism of R onto either R/F(R) or R/(pR), where p is a prime. This remark and the next lemma are used in conjunction with Amitsur's Lemma, in what follows.

LEMMA 5. *If the nilpotent elements of* R *form an ideal, then so do those of* R/F(R).

Proof. Let $\bar{x}$, $\bar{y}$, and $\bar{r}$ be in R/F(R), with $\bar{x}$ and $\bar{y}$ nilpotent, say $\bar{x}^n = \bar{y}^n = 0$ with $n \geq 1$. We shall show that $\bar{x}\bar{r}$, $\bar{r}\bar{x}$, and $\bar{x} - \bar{y}$ are nilpotent. Since x^n and y^n are in F(R), $mx^n = my^n = 0$ for some positive integer m. Thus

$$0 = m^n x^n = (mx)^n \quad \text{and} \quad 0 = m^n y^n = (my)^n,$$

whence mx and my (and therefore mx - my) are nilpotent. Since $(mx - my)^t = 0$ $(t \geq 1)$, we deduce that $m^t(x - y)^t = 0$, so that $(x - y)^t$ is in F(R), whence $(\bar{x} - \bar{y})^t = 0$.

Since mx is nilpotent, mxr is also nilpotent. Hence there exists a positive integer k such that $(mxr)^k = 0$, in other words, $m^k(xr)^k = 0$. Thus $(xr)^k$ is in F(R), whence $(\bar{x}\bar{r})^k = 0$. Similarly, $\bar{r}\bar{x}$ is nilpotent.

THEOREM 2. *Let* R *be a* K-*ring whose nilpotent elements form an ideal. If* R *satisfies either*

(a) k *is independent of* x *or*

(b) k *and* n *are independent of* y,

then C(R) *is a nil ideal.*

Proof. Let $\mathscr{R}$ be the class of all rings satisfying the hypotheses of the theorem, and for R in $\mathscr{R}$ let Q(R) = C(R). That Q(R) is nil whenever R in $\mathscr{R}$ has characteristic 0 follows from Theorem 1. Also, since Drazin has proved that every K-ring of prime characteristic has nil commutator ideal [1], it follows that hypothesis (ii) of Lemma 4 is satisfied. By the remark following Lemma 4, it remains to show that if R is in $\mathscr{R}$ and θ is the natural homomorphism of R onto either R/F(R) or R/(pR) (p a prime), then $R\theta$ is in $\mathscr{R}$. Thus all that really needs verification is that if θ is either of these homomorphisms, then the nilpotent elements of $R\theta$ form an ideal. Now if θ is onto R/F(R), Lemma 5 applies, whereas if θ is onto R/(pR), then $C(R\theta)$ is nil since $R\theta$ is a K-ring of prime characteristic, and it follows from this that the nilpotent elements of $R\theta$ form an ideal. Hence $\mathscr{R}$ satisfies all the hypotheses of Lemma 4, and the theorem is proved.

The next lemma is motivated by the technique used in the proof of the well-known theorem of Kaplansky, Herstein, and Kleinfeld which states that if R is a ring for which there exists a fixed positive integer n such that $(xy - yx)^n = 0$ for all x, y in R, then the nilpotent elements of R form an ideal [7], [5, p. 29].

LEMMA 6. *If* x *in* R *satisfies* $x^2 = 0$ *and if there exists a positive integer* n(x) *such that* $(yx)^{n(x)} = 0$ *for every* y *in* R, *then the ideal generated by* x *in* R *is locally nilpotent.*

Proof. Let T be the ideal generated by x in R. For every integer r and every element y in R, the formulas

$$(rx + yx)^{j+1} = rx(yx)^j + (yx)^{j+1} \qquad (j = 0, 1, \cdots)$$

are easily proved by induction on j. Hence $(rx + yx)^{n(x)+1} = 0$. Thus, by a theorem of Levitzki, the left ideal generated by x is a locally nilpotent ring [9], [5, p. 28]. Hence, x is in L(R) and therefore $T \subset L(R)$; that is, T is locally nilpotent.

The following lemma replaces Theorem 2.1 of [1], the proof of which is in error.

LEMMA 7. *Let* R *be an* m-*ring with* k *and* n *independent of* y. *If* x *in* R *is nilpotent, then* x^{m^2} *is in* L(R).

Proof. First suppose that an element z in R satisfies $z^{m+1} = 0$. By hypothesis there exist integers k = k(z) and n = n(z) such that to each y in R there correspond integers s and t $(1 \leq t \leq m)$ and an element a in R with

$$z^{m-t}e_k(sz^{t+1} + z^{t+1}a - z^t, (yz^m)^n) = 0.$$

But then

$$0 = z^{m-t} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (yz^m)^{nj} (sz^{t+1} + z^{t+1}a - z^t)(yz^m)^{n(k-j)}$$

$$= z^{m-t}(sz^{t+1} + z^{t+1}a - z^t)(yz^m)^{nk}$$

$$= -z^m (yz^m)^{nk}.$$

Thus for every y in R, $(yz^m)^{nk+1} = 0$, where the exponent $nk+1$ depends only on z. Also, $(z^m)^2 = 0$, and so, by Lemma 6, the ideal generated by z^m (and hence z^m itself) is contained in L(R). Hence, if $z^{m+1} = 0$, then z^m is in L(R).

Now suppose x in R is nilpotent.

First assume L(R) = (0). If $x^m = 0$, then $x^{m^2} = 0$ is in L(R). Now assume that $x^m \neq 0$. There exists an integer $r > 1$ such that

$$x^{mr} = 0 \quad \text{and} \quad x^{m(r-1)} \neq 0.$$

If $m < r$, then $(x^{r-1})^{m+1} = x^{mr}x^{r-m-1} = 0$, and hence, by what was proved above, taking $z = x^{r-1}$, we deduce that $(x^{r-1})^m$ is in L(R) = (0), a contradiction. Thus $r \leq m$ and $x^{m^2} = 0$.

Now assume $L(R) \neq (0)$. Since x is nilpotent in R, $\bar{x}$ is nilpotent in R/L(R), so that $\bar{x}^{m^2} = 0$ by the above case. Therefore x^{m^2} is in L(R).

COROLLARY. *If* R *is a* 1*-ring (or in particular any* K*-ring) with* k *and* n *independent of* y*, then the nilpotent elements of* R *form an ideal and* N(R) = L(R).

The next theorem is Theorem 6.2 of [1]; the proof given there is invalid.

THEOREM 3. *If* R *is a* K*-ring with* k *and* n *independent of* y*, then* C(R) *is a nil ideal.*

Proof. By the corollary to Lemma 7 the nilpotent elements of R form an ideal. Hence Theorem 2(b) applies.

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The University of Texas