CHARACTERISTIC NUMBERS AND HOMOTOPY TYPE

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let Ω denote the oriented cobordism ring (see [10]), and [M] the oriented cobordism class of the C^∞-manifold M, which we assume to be closed, compact, and oriented, but not necessarily connected. Ω is graded by manifold dimension. In Ω_n, let I_n denote the set of all classes of the form [M] - [M'], where M and M' are n-manifolds of the same oriented homotopy type. It is clear that I_n is a subgroup of Ω_n and that the graded group $I = (I_0, I_1, I_2, \cdots)$ is an ideal in Ω.

The following result follows easily from the definitions and from certain elementary facts about Pontrjagin and Stiefel-Whitney numbers.

THEOREM 1. *I_n is a free abelian group. If* $n \not\equiv 0 \pmod 4$, *then* $I_n = 0$. *If* $n \equiv 0 \pmod 4$, *then* co-rank $I_n \geq 1$, *where* co-rank I_n = rank (Ω_n/I_n).

Note that since $\Omega_4 \simeq Z$ (this is well-known), Theorem 1 implies that $I_4 \simeq 0$.

Atiyah and Hirzebruch prove, in [1], that Pontrjagin classes are homotopy invariants (mod 8). We use this to prove the following assertion.

THEOREM 2. *The members of* I_n *are divisible by* 8.

The results of [5]—see the proof of Theorem 3 in Section 3, below—imply the following.

THEOREM 3. $\Omega \otimes Q \simeq Q[Y_4] \oplus (I \otimes Q)$.

(Explanation of notation: Q denotes the field of rational numbers, $Q[Y_4]$ denotes the polynomial ring over Q generated by some $Y_4 \in \Omega_4 \otimes Q$, and the symbol $\oplus$ denotes vector-space direct sum.)

COROLLARY 3.1. $I \otimes Q$ *is a prime ideal in* $\Omega \otimes Q$.

COROLLARY 3.2. co-rank $I_{4k} = 1$.

COROLLARY 3.2.1. *There is, up to a rational multiple, only one homotopy-invariant rational linear combination of Pontrjagin numbers (the* L_k*-genus* (see [4, p. 13]), *being such a combination).*

In [9], Tamura constructs certain 8-manifolds representing nontrivial elements of I_8; in [5], we extend his results to dimension 12. This enables us to obtain partial information about generators for I_8 and I_{12}.

THEOREM 4. *Let* X_i *denote the class in* Ω_{4i} *of complex projective* 2i*-space* (i = 1, 2, 3), *and let* $A = X_2 - X_1^2$, $B = X_3 - X_2X_1$. *Then*

(i) I_8 *is generated by* $2^n \cdot 48A$, *for some integer* n $(0 \leq n \leq 3)$, *and*

(ii) I_{12} *has rank* 2 *and contains* $384X_1A$ *and* 576B; *all elements of* I_{12} *are of the form* $rX_1A + sB$, *where* $r \equiv 0 \pmod{24}$ *and* $s \equiv 0 \pmod{72}$.

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In Section 3 we prove Theorems 1, 2, and 4 and derive all the corollaries. We also derive Theorem 3 from the results of [5].

The main results of this paper that do not depend on [5] are proved in Section 4. To state them, we need some notation.

Let $\Pi(k)$ denote the set of all partitions of the integer k. For any odd prime p, let $\Pi(p, k)$ be the subset of $\Pi(k)$ consisting of all partitions that contain no integers of the form $(p^j - 1)/2$. Finally, for any pair (p, k) satisfying

(i) p is an odd prime, and

(ii) $2k \equiv 0 \pmod{p-1}$,

let $\Pi'(p, k) \subset \Pi(p, k)$ consist of all partitions containing only multiples of $(p-1)/2$. Let $\pi(k)$ (respectively, $\pi(p, k)$ and $\pi'(p, k)$) denote the cardinality of $\Pi(k)$ (respectively, of $\Pi(p, k)$ and $\Pi'(p, k)$).

THEOREM 5. *If* $2k \equiv 0 \pmod{p-1}$, *then* $\text{co-dim}_{Z_p}(I_{4k} \otimes Z_p) \geq \pi'(p, k)$.

For (p, k) defined as above, let $m_{p,k}$ denote the maximum number of linearly independent, homotopy-invariant, (mod p)-characteristic numbers. Then the proof of Theorem 5 establishes a further result:

THEOREM 6. $m_{p,k} \geq \pi(k) + \pi'(p, k) - \pi(p, k)$.

CONJECTURE. *Equality holds in Theorems* 5 *and* 6.

This conjecture, if it is true, implies that certain (mod p)-characteristic classes defined in [6, p. 120] (first considered by Wu in [12]), together with the relations among (mod p)-characteristic classes, computed in [3] and partially computed in [2], generate *all* the homotopy-invariant (mod p)-characteristic numbers.

The following problems, still unsolved, are listed in what I think is increasing order of difficulty.

(1) What is the p-primary component of Ω_{4k}/I_{4k} $(p > 2)$? Theorem 5 gives only a partial answer to this question.

(2) What is the 2-primary component of Ω_{4k}/I_{4k}? By Theorem 2 we know that Ω_{4k}/I_{4k} contains elements of order 8.

(3) What is the ring structure of Ω/I?

(4) Construct generators for I.

2. SOME KNOWN RESULTS AND DEFINITIONS

The reader who is familiar with the results and notation of [2] and of [6] or [10] may, after a quick glance at the notational conventions, skip to Sections 3 and 4.

Let BSO denote the classifying space for the stable group SO. It is well known that $H^*(BSO; Q)$ is a polynomial ring over Q generated by the universal Pontrjagin classes, $p_i \in H^{4i}(BSO; Q)$. In another well-known characterization, certain classes $s(\omega) \in H^{4i}(BSO; Q)$—with ω ranging over $\Pi(i)$ for all i—are shown to form a vector-space basis over Q of $H^*(BSO; Q)$. We may describe the classes $s(\omega)$ as follows: suppose $\omega \in \Pi(i)$; let $\sigma_1, \cdots, \sigma_i$ denote the elementary symmetric polynomials in indeterminates $t_1, \cdots, t_i$; if $\omega = (i_1, \cdots, i_r)$ $(r \leq i)$, let $S_\omega(\sigma_1, \cdots, \sigma_i)$ be the polynomial in $\sigma_1, \cdots, \sigma_r$ expressing the symmetric polynomial with fewest monomials containing the term $t_1^{i_1} \cdots t_r^{i_r}$; let $s(\omega) \in H^{4i}(BSO; Q)$ denote the polynomial $S_\omega(p_1, \cdots, p_i)$; it is easy to show that $s(\omega)$ is an integral linear combination of monomials $p_{j_1} \cdots p_{j_s}$ $(1 \leq j_\alpha \leq i)$, where $(j_1, \cdots, j_s) \in \Pi(i)$.

We now relate $H^*(BSO; Q)$ to Ω. Let M be any closed, compact, oriented, connected C^∞-manifold, and let t_M: $M \to BSO$ be the classifying map for the stable tangent bundle. Then we denote by $p_{j_1} \cdots p_{j_s}(M)$ the class $t_M^*(p_{j_1} \cdots p_{j_s})$ in $H^{4i}(M; Q)$, where $(j_1 \cdots j_s) \in \Pi(i)$. If dimension $M = 4k$ and $(j_1, \cdots, j_s) \in \Pi(k)$, we let $p_{j_1} \cdots p_{j_s}[M] \in Q$ denote the evaluation of $p_{j_1} \cdots p_{j_s}(M)$ on the orientation class of M. The numbers $p_{j_1} \cdots p_{j_s}[M]$ are actually integers. As is shown by Pontrjagin, they are invariants of cobordism class. Hence, there is defined a graded vector space "evaluation" homomorphism

$$e_Q: H^*(BSO; Q) \to \mathrm{Hom}_Q(\Omega \otimes Q, Q).$$

One of the chief results of [10] is that e_Q is an isomorphism. This implies that Ω_{4k} is an abelian group of rank $\Pi(k)$ and that Ω_n is a torsion group, for $n \not\equiv 0 \pmod 4$.

We can describe the result more fully by stating some of the properties of the numbers $s(\omega)[M]$, where

$$\text{dimension } M = 4k \quad \text{and} \quad \omega \in \Pi(k).$$

Clearly, the numbers $s(\omega)[M]$, being integral linear combinations of the numbers $p_{j_1} \cdots p_{j_s}[M]$, are integers. Given $\omega \in \Pi(k)$ and $\omega' \in \Pi(k')$, we define their product by juxtaposition. That is, if $\omega = (i_1, \cdots, i_r)$ and $\omega' = (i_1', \cdots, i_s')$, then

$$\omega\omega' = \omega'\omega = (i_1, \cdots, i_r, i_1', \cdots, i_s').$$

We shall say that a partition ω is a refinement of a partition $\omega' = (i_1', \cdots, i_s')$, written $\omega \geq \omega'$, if $\omega = \omega_1 \cdots \omega_s$, where $\omega_\alpha \in \Pi(i_\alpha')$ $(\alpha = 1, \cdots, s)$.

PROPOSITION 2.1 (Thom [10]).

$$s(\omega)[M_1 \times \cdots \times M_r] = \sum_{\omega_1 \cdots \omega_r = \omega} s(\omega_1)[M_1] \cdots s(\omega_r)[M_r].$$

This proposition has two important corollaries, also due to Thom, the proofs of which can be found in [6] and in [10]:

COROLLARY 2.1.1. *Let* dim $M_\alpha = 4i_\alpha$ *for* $\alpha = 1, \cdots, r$. *Then*

$$s(\omega)[M_1 \times \cdots \times M_r] = \begin{cases} 0 & \text{if } \omega \not\geq (i_1, \cdots, i_r), \\ s(i_1)[M_1] \cdots s(i_r)[M_r] & \text{if } \omega = (i_1, \cdots, i_r). \end{cases}$$

COROLLARY 2.1.2. *Let* $\{M_i\}$ *be a sequence of manifolds such that* dim $M_i = 4i$. *Let* Γ *be the subring of* $\Omega \otimes Q$ *generated by the* $[M_i]$. *Then the* $[M_i]$ *are independent ring generators of* Γ *if and only if* $s(i)[M_i] \neq 0$ $(i = 1, 2, \cdots)$.

Since such a sequence clearly exists (for example, let M_i be complex projective 2i-space), $\Omega \otimes Q$ contains a graded polynomial ring Γ with one generator of each degree $4i$ $(i = 1, 2, \cdots)$.

Since e_Q: $H^*(BSO; Q) \to \Omega \otimes Q$ is a graded vector space isomorphism and Γ and $H^*(BSO; Q)$ are isomorphic as Q-algebras, it is easy to see that $\Omega \otimes Q = \Gamma$.

Milnor [8] computes the odd torsion of Ω. To describe a key step in his computation, we introduce the following notation.

Let $\mathscr{A}$ denote the (mod p)-Steenrod algebra, p being an odd prime. Let $\beta \in \mathscr{A}$ denote the Bockstein coboundary operator, and (β) the two-sided ideal generated by β. Milnor constructs the stable object MSO, a stable counterpart to the Thom space MSO_n of the universal bundle over BSO_n. Moreover, the Thom isomorphisms

$$\phi_n: H^*(BSO_n; Z_p) \to H^*(MSO_n; Z_p)$$

induce an isomorphism

$$\phi: H^*(BSO; Z_p) \to H^*(MSO; Z_p).$$

We can now state Milnor's result as follows.

PROPOSITION 2.2. $H^*(MSO; Z_p)$ *is a free* $\mathscr{A}/(B)$*-module on generators* $\phi s(\lambda)$, *where* λ *ranges over* $\Pi(p, k)$ *for all* k.

From this proposition, Milnor deduces the following result.

COROLLARY 2.2.1. Ω *has no odd torsion.*

Milnor also characterizes the ring structure of Ω modulo 2-torsion (see [7]), as follows:

PROPOSITION 2.3. Ω *modulo 2-torsion is a polynomial ring on generators* $Y_k \in \Omega_{4k}$. *The* Y_k *are determined by the following necessary and sufficient condition:*

$$s(k)(Y_k) = \begin{cases} \pm p & \textit{if } 2k+1 = p^j, \\ \pm 1 & \textit{otherwise.} \end{cases}$$

Finally, Wall [11] proves that the torsion elements of Ω are determined by their Stiefel-Whitney numbers. It follows that Ω_n is the direct sum of certain numbers of copies of the group Z and Z_2.

For any odd prime p, we can define the graded vector space homomorphism

$$e_p: H^*(BSO; Z_p) \to \mathrm{Hom}_{Z_p}(\Omega \otimes Z_p, Z_p),$$

the definition being completely analogous to that of e_Q. The structure of $H^*(BSO; Z_p)$ is similar to that of $H^*(BSO; Q)$, the only difference being the coefficient field. Hence, $H^*(BSO; Z_p)$ is a polynomial ring on the universal (mod p)-Pontrjagin classes, still denoted by $p_1, p_2, \cdots$. Classes $s(\omega) \in H^{4i}(BSO; Z_p)$ are defined as before and form a vector-space basis of $H^*(BSO; Z_p)$. Hence, $H^*(BSO; Z_p)$ and $\mathrm{Hom}_{Z_p}(\Omega \otimes Z_p, Z_p)$ are vector-space-isomorphic. However, the homomorphism e_p is not an isomorphism. The kernel of e_p has been computed in [2] and in [3]. We use the terminology of [2] to describe the result.

For any topological space X, let $H^{**}(X; Z_p)$ be the direct product of the singular cohomology groups $H^i(X; Z_p)$ $(i = 0, 1, 2, \cdots)$. We can think of the direct sum $H^*(X; Z_p)$ as being included in $H^{**}(X; Z_p)$. The cup-product operation gives both H^{**} and H^* a skew-commutative ring structure. If $x \in H^{**}(X; Z_p)$, let $(x)_i$ denote its component in $H^i(X; Z_p)$. It is easy to show (by inductive definition, for example), that if $(x)_0 \neq 0$, then x is invertible in H^{**}.

Letting P^i $(i = 0, 1, 2, \cdots)$ denote the Steenrod reduced powers with respect to the prime p, we can regard $P = P^0 + P^1 + P^2 + \cdots$ as an automorphism of rings $H^{**}(X; Z_p) \to H^{**}(X; Z_p)$. Clearly, $P(H^*(X; Z_p)) = H^*(X; Z_p)$.

We define

$$Wu(P) = P^{-1} \phi^{-1} P\phi(1) \in H^{**}(BSO; Z_p),$$

where ϕ is the extension to $H^{**}(BSO; Z_p)$ of the Thom isomorphism described above.

PROPOSITION 2.4. $(\ker e_p)_{4k} = \{(Py - y\,Wu(P)_{4k} \mid y \in H^{**}(BSO; Z_p)\}$.

We express this result in a more convenient form. To do so, we still follow [2] and define an automorphism D of $H^{**}(BSO; Z_p)$ such that $D^2 = 1$. The class $1 + p_1 + p_2 + \cdots = p$ is invertible. Let $D(p_i) = (p^{-1})_i$, and extend D to the entire ring by requiring that it be a ring homomorphism. Note that D preserves degree. It is proved in [2] that if $\{K_i\}$ is any multiplicative sequence (see [4]) and $K = 1 + K_1 + K_2 + \cdots \in H^{**}(BSO; Z_p)$, then $D(K) = K^{-1}$.

PROPOSITION 2.4.1. $\phi D(\ker e_p)_{4k} = \{(P\phi(y))_{4k} \mid y \in \sum_{i=0}^{k-1} H^{4i}(BSO; Z_p)\}$.

This result is proved in [2].

We remind the reader of one more result. If we replace P by Sq, we can define classes

$$Wu(Sq) = Sq^{-1} \phi^{-1} Sq\, \phi(1) \in H^{**}(BO; Z_2).$$

The class $Sq\,Wu(Sq) \in H^{**}(BO; Z_2)$ is the direct product of the universal Stiefel-Whitney classes w_i. Let $q_i = (PWu(P))_i$.

PROPOSITION 2.5. (Wu [12]). (i) *The classes* w_i *are homotopy invariants.* (ii) *The classes* q_i *are oriented homotopy invariants.*

(This is the only time that we distinguish between homotopy and orientation-preserving homotopy, the latter being understood elsewhere.)

It is now clear that all characteristic numbers obtained from the w_i or from the q_i are homotopy invariants.

3. PROOFS OF THEOREMS 1 TO 4 AND COROLLARIES 3.1, 3.2, AND 3.2.1

Proof of Theorem 1. Each torsion element of Ω_n (hence each torsion element of I_n) is determined by its Stiefel-Whitney numbers. Since, according to Proposition 2.5, these are homotopy invariants, they vanish on I_n. Hence, I_n has no torsion and is free.

If $n \not\equiv 0 \pmod 4$, then rank $\Omega_n = 0$ according to Corollary 2.1.2, so that $I_n = 0$.

If $n = 4k$, then it suffices to show that the vector space $\Omega_n \otimes Q/I_n \otimes Q$ is non-trivial. Now $\Omega_n \otimes Q/I_n \otimes Q$ is isomorphic to the subspace of $\mathrm{Hom}_Q(\Omega_n \otimes Q, Q)$ that annihilates $I_n \otimes Q$. But e_Q^{-1} (annihilator $(I_n \otimes Q)$) can be described as the subspace of all homotopy-invariant rational linear combinations of Pontrjagin numbers, and, according to Hirzebruch, this latter subspace is not empty (it contains the L_k-genus). Q. E. D.

Proof of Theorem 2. Since I_n is free, we may suppose that $I \subset \Omega$ modulo 2-torsion. Let $\{Y_i\}$ be a sequence of generators for Ω modulo 2-torsion. If ω denotes any partition $(i_1, \cdots, i_r)$, let $Y_\omega = Y_{i_1} \cdots Y_{i_r}$. Recall that $s(\omega')(Y_\omega) = 0$ unless $\omega' \geq \omega$. Moreover, $s(\omega)(Y_\omega) = s(i_1)(Y_{i_1}) \cdots s(i_r)(Y_{i_r})$, which, according to Proposition 2.3, is always odd.

We make use of the result of [1] that Pontrjagin classes—and, hence, the numbers $s(\omega)$—are homotopy invariants (mod 8).

Fix k, and choose any $\sigma \in I_{4k}$. According to the above conventions, we may write

$$\sigma = \sum_{\omega \in \Pi(k)} a_\omega Y_\omega .$$

We prove that $a_\omega \equiv 0 \pmod 8$ for all $\omega \in \Pi(k)$. We proceed by induction, using the refinement-ordering $\leq$ defined in Section 2. Note that $s(\omega)(\sigma) \equiv 0 \pmod 8$, for all $\omega \in \Pi(k)$.

Applying $s(k)$ to both sides of the above equation, we obtain the relation

$$s(k)(\sigma) = \sum_{\omega \in \Pi(k)} a_\omega \, s(k)(Y_\omega) .$$

Since $(k) \not\geq \omega$ for all $\omega \neq (k)$, $s(k)(Y_\omega) = 0$ for these ω. Hence,

$$s(k)(\sigma) = a_k \, s(k)(Y_k) .$$

Since $s(k)(Y_k)$ is odd and $s(k)(\sigma) \equiv 0 \pmod 8$, we obtain $a_k \equiv 0 \pmod 8$. This completes the first step of the induction.

Choose any $\omega_0 \in \Pi(k)$ and assume that $a_\omega \equiv 0 \pmod 8$ for all $\omega < \omega_0$. Then, write

$$\sigma - \sum_{\substack{\omega \in \Pi(k) \\ \omega < \omega_0}} a_\omega Y_\omega = \sum_{\substack{\omega \in \Pi(k) \\ \omega_0 \not\geq \omega}} a_\omega Y_\omega + a_{\omega_0} Y_{\omega_0} .$$

Now apply $s(\omega_0)$ to both sides. Since the left side of the resulting equation is congruent to 0 (mod 8), we see that

$$a_{\omega_0} s(\omega_0)(Y_{\omega_0}) + \sum_{\substack{\omega \in \Pi(k) \\ \omega_0 \not\geq \omega}} a_\omega \, s(\omega_0)(Y_\omega) \equiv 0 \pmod 8 .$$

Since $s(\omega_0)(Y_\omega) = 0$ when $\omega_0 \not\geq \omega$, we get the relation $a_{\omega_0} s(\omega_0)(Y_{\omega_0}) \equiv 0 \pmod 8$, so that, since $s(\omega_0)(Y_{\omega_0})$ is odd, $a_{\omega_0} \equiv 0 \pmod 8$. Q.E.D.

Proof of Theorem 3. In [5], we construct elements $Z_k \in I_{4k}$ such that $s(k)(Z_k) \neq 0$ $(k > 1)$. Choosing any $Y_1 \in \Omega_4$ that satisfies $s(1)(Y_1) \neq 0$, and using Corollary 2.1.2, we obtain the isomorphism

$$\Omega \otimes Q \simeq Q[Y_1, Z_2, Z_3, \cdots].$$

It now suffices to show that the ideal $(Z_2, Z_3, \cdots)$ is identical with $I \otimes Q$. Clearly, $(Z_2, \cdots) \subset I \otimes Q$. Moreover,

$$\text{co-dim}\,(Z_2, \cdots)_{4k} \leq 1 \leq \text{co-dim}\, I_{4k} \otimes Q,$$

by Theorem 1. Hence, we have equality. Q.E.D.

Proof of Corollary 3.1. Theorem 3 implies that $\Omega \otimes Q/I \otimes Q \simeq Q[Y_4]$. Since $Q[Y_4]$ is an integral domain, the result follows.

Proof of Corollary 3.2.

$$\text{co-rank}\, I_{4k} = \dim_Q \Omega_{4k} \otimes Q/I_{4k} \otimes Q = \dim_Q (Q[Y_4])_{4k} = 1. \quad \text{Q.E.D.}$$

Proof of Corollary 3.2.1.

$$\dim_Q (\text{annihilator of } I_{4k} \otimes Q) = \text{co-rank}\, I_{4k} = 1. \quad \text{Q.E.D.}$$

Proof of Theorem 4. We divide the proof into several steps.

(i) A tedious but elementary computation of Pontrjagin numbers shows that the elements $Z_i \in I_{4i}$ $(i = 2, 3)$ constructed in [9] and [5], respectively, and used in the proof of Theorem 3, are, respectively, $384A = 2^3 \cdot 48A$ and $576B + 384kX_1 A$, for some large integer k. (Actually, the smallest such k is 293 423 189 379.) Since I is an ideal, $384X_1 A \in I_{12}$, so that $576B \in I_{12}$.

(ii) According to Theorem 3, A generates $I_8 \otimes Q$, and $X_1 A$ and B generate $I_{12} \otimes Q$. It follows easily from the known structure of Ω_8 and Ω_{12} that some integral multiple of A (respectively, some integral linear combinations of $X_1 A$ and B) generates I_8 (generate I_{12}). Step (i) provides "upper bounds" for possible generators. In the next two lemmas we establish lower bounds. This will complete the proof of Theorem 4.

LEMMA 3.1. *The elements of* I_8 *are divisible by* 48.

Proof. According to [1], Pontrjagin classes are homotopy invariants (mod 24). Hence, the number p_1^2 is a homotopy invariant (mod 48). Elementary calculation shows that $s(2) = p_1^2 - 2p_2$, so that $s(2)$ numbers are homotopy invariants (mod 48). Suppose that $kA \in I_8$. It is well known that $s(2)(A) = 5$. Hence, $5k \equiv 0 \pmod{48}$, whence $k \equiv 0 \pmod{48}$. Q.E.D.

LEMMA 3.2. *If* $r(X_1 A) + sB \in I_{12}$, *then* $r \equiv 0 \pmod{24}$ *and* $s \equiv 0 \pmod{72}$.

Proof. We tabulate some well-known characteristic numbers.

	$X_1 A$	B
$s(3)$	0	7
p_1^3	63	118
$p_1 p_2$	26	42

By a method similar to that in the previous lemma, and using the fact that the p_i are homotopy invariants (mod 24), we can show that

$$s_3(\sigma) \equiv 0 \pmod{2^3 \cdot 3^2 \cdot 7} \text{ for } \sigma \in I_{12}.$$

Moreover, we can obtain the relations

$$p_1^3(\sigma) \equiv 0 \pmod{2^3 \cdot 3^2} \quad \text{and} \quad p_1 p_2(\sigma) \equiv 0 \pmod{2^3 \cdot 3} \quad \text{for } \sigma \in I_{12}.$$

Now let $\sigma = rX_1 A + sB \in I_{12}$. Then, using the above table and congruences, we deduce that

$$7s \equiv 0 \pmod{2^3 \cdot 3^2 \cdot 7},$$

$$63r + 118s \equiv 0 \pmod{2^3 \cdot 3^2},$$

$$26r + 42s \equiv 0 \pmod{2^3 \cdot 3}.$$

The desired congruences follow from these.

4. PROOFS OF THEOREMS 5 AND 6

We shall prove Theorems 5 and 6 by defining a subring $V^{**} \subset H^{**}(BSO; Z_p)$ such that

$$e_p(V^{4k}) \subset \text{annihilator}(I_{4k} \otimes Z_p),$$

and by using the following proposition, which we shall establish below.

PROPOSITION 4.1. *If* $2k \equiv 0 \pmod{p-1}$, *then* $\dim_{Z_p} e_p(V^{4k}) = \pi'(p, k)$.

Theorem 5 follows immediately from Proposition 4.1. To obtain Theorem 6, let $M_{p,k} \subset H^{4k}(BSO; Z_p)$ denote the subspace of all homotopy invariant characteristic numbers. That is, let $M_{p,k} = e_p^{-1}(\text{annihilator}(I_{4k} \otimes Z_p))$. Recall that we defined $m_{p,k}$ as the dimension of $M_{p,k}$. Clearly, $V^{4k} + (\ker e_p)_{4k} \subset M_{p,k}$. Results of Milnor [8] imply that $\dim_{Z_p} (\ker e_p)_{4k} = \pi(k) - \pi(p, k)$. These comments, together with Proposition 4.1, yield Theorem 6. It remains to define V^{**} and to compute $\dim_{Z_p} e_p(V^{4k})$ for $2k \equiv 0 \pmod{p-1}$.

Definition of V^{**}. Let $q_i = (PWu(P))_i$, as in Section 2; let V^* be the subring of $H^*(BSO; Z_p)$ generated by the q_i, and let V^{**} be the direct product of the V^i. According to [2, p. 170],

$$q_i = \underbrace{s((p-1)/2, \cdots, (p-1)/2)}_{i \text{ tuple}}.$$

That is, q_i may be considered as the ith elementary symmetric polynomial in indeterminates $t_1^{(p-1)/2}, \cdots$. Hence, $V^* \subset H^*(BSO; Z_p)$ is the vector subspace (over Z_p) spanned by those $s(\omega)$ for which ω consists only of multiples of $(p-1)/2$.

LEMMA 4.2. $e_p(V^{4k}) \subset$ annihilator $(I_{4k} \otimes Z_p)$ *and* $P\phi(V^{**}) \subset \phi(V^{**})$, *where* P *is the Steenrod power automorphism and* ϕ *is the Thom isomorphism, both described in Section 2.*

Proof. The first statement follows immediately from Proposition 2.5.

To prove the second statement, we fix a positive integer k and consider $H^*(BSO_{2n+1}; Z_p)$ and $H^*(MSO_{2n+1}; Z_p)$, for n large relative to 4k. It will be convenient (and correct) to think of the former as the ring of all symmetric polynomials in indeterminates $s_1^2, \cdots, s_n^2$ (see [6]). The subring W^* of all symmetric polynomials in $s_1^{p-1}, \cdots, s_n^{p-1}$ is precisely the ring for which V^* is the limiting case. It will suffice to show that $P\phi_{2n+1}(W^{4k}) \subset \phi_{2n+1}(W^*)$.

The action of ϕ_{2n+1} can be described by the equation $\phi_{2n+1}(a) = s_1 \cdots s_n a$ (see [8, p. 517]). Now let $x \in \phi_{2n+1}(W^{4k})$. Then x is a linear combination of monomials of the form

$$s_1^{i_1(p-1)+1} \cdots s_r^{i_r(p-1)+1} s_{r+1} \cdots s_n.$$

It is well known that $P(s_i) = s_i + s_i^p$. Hence,

$$P(s_1^{i_1(p-1)+1} \cdots s_r^{i_r(p-1)+1} s_{r+1} \cdots s_n)$$

$$= s_1 \cdots s_n s_1^{\alpha_1} \cdots s_r^{\alpha_r}(1+s_1^{p-1})^{\beta_1} \cdots (1+s_r^{p-1})^{\beta_r}(1+s_{r+1}^{p-1}) \cdots (1+s_n^{p-1}),$$

where $\alpha_\ell = i_\ell(p-1)$, $\beta_\ell = i_\ell(p-1)+1$ ($\ell = 1, \cdots, r$). Clearly, the right side of this equation is the sum of monomials of the same form as

$$s_1^{i_1(p-1)+1} \cdots s_r^{i_r(p-1)+1} s_{r+1} \cdots s_n.$$

It follows that $P(x) \in \phi_{2n+1}(W^*)$. Q.E.D.

LEMMA 4.3. $\phi(V^*)$ *is a free* $\mathscr{A}/(\beta)$-*module on generators* $\phi s(\lambda)$, *where* λ *ranges over* $\Pi'(p, k)$ *for all* k *satisfying the condition* $2k \equiv 0 \pmod{p-1}$.

Proof. In [8], Milnor describes a Z_p-basis $\{P^R\}$ for $\mathscr{A}/(\beta)$, where R ranges over all finite sequences of nonnegative integers (two sequences being considered equal if they are equal up to their last positive term), and where P^R is a certain polynomial in the P^i. By the second part of Lemma 4.2, therefore, $\phi(V^*)$ is an $\mathscr{A}/(\beta)$-module. Moreover, it follows from Proposition 2.2, that the elements $\phi s(\lambda)$ ($\lambda \in \Pi'(p, k)$) are free over $\mathscr{A}/(\beta)$. It remains to show that they generate $\phi(V^*)$ over $\mathscr{A}/(\beta)$.

Milnor shows that corresponding to each partition ω, there exists a unique $\lambda \in \Pi(p, k)$, for some k, and a unique P^R such that

$$P^R(\phi s(\lambda)) = \phi s(\omega) + \sum_{\omega' < \omega} a_{\omega'} \phi s(\omega'),$$

where $<$ is a certain complicated ordering. Moreover, λ is obtained from ω by deletion of all the integers $(p^j - 1)/2$ $(j > 0)$.

Suppose ω consists only of multiples of $(p - 1)/2$. Then the same is true of the corresponding λ. Hence, since $\phi(V^*)$ is an $\mathscr{A}/(\beta)$-module, $P^R(\phi s(\lambda)) \in \phi(V^*)$. Therefore the sum

$$\sum_{\omega' < \omega} a_{\omega'} \phi s(\omega') = P^R(\phi s(\lambda)) - \phi s(\omega)$$

belongs to $\phi(V^*)$. Now, $\phi(V^*)$ has as a Z_p-basis all $\phi s(\omega)$ for which ω consists only of multiples of $(p - 1)/2$. Since the entire collection of $\phi s(\omega)$ (ω arbitrary) is linearly independent over Z_p, we deduce that each of the ω' appearing in the sum above contains only multiples of $(p - 1)/2$. Therefore, using the equation in the preceding paragraph, and the ordering $<$, we can prove inductively that for every partition ω consisting only of multiples of $(p - 1)/2$,

$$\phi s(\omega) = \sum a_{\lambda'} P^{R'} \phi s(\lambda'),$$

where λ' ranges over $\Pi'(p, k)$ for all k satisfying the relation $2k \equiv 0 \pmod{p - 1}$. Clearly, almost all of the $a_{\lambda'}$ are equal to zero. Since these $\phi s(\omega)$ generate $\phi(V^*)$ over Z_p, the $\phi s(\lambda')$ generate $\phi(V^*)$ over $\mathscr{A}/(\beta)$. Q. E. D.

Recall that $D: H^{**}(BSO; Z_p) \to H^{**}(BSO; Z_p)$ is a certain automorphism of period 2 (see Section 2).

LEMMA 4.4. $D(V^{**}) = V^{**}$.

Proof. According to Wu [12] the sequence $\{q_i\}$ is the multiplicative sequence corresponding to the power series $1 + t^{(p-1)/2}$. Hence, if $q = 1 + q_1 + \cdots$, then $D(q) = q^{-1}$ (see Section 2). It is easy to see that

$$(q^{-1})_i = -q_i - \sum_{j=1}^{i-1} (q^{-1})_j \, q_{i-j}.$$

Hence, by induction, $(q^{-1})_i \in V^{2i(p-1)}$, that is, $D(q_i) \in V^*$. Therefore, $D(V^{**}) \subset V^{**}$, and since $D^2 = 1$, $V^{**} \subset D(V^{**})$. Q. E. D.

Proof of Proposition 4.1. Consider the exact sequence

$$0 \to (\ker e_p)_{4k} \cap V^{4k} \to V^{4k} \xrightarrow{e_p} e_p(V^{4k}) \to 0.$$

Applying the isomorphism ϕD to the second and third terms of the sequence, we obtain an exact sequence

$$0 \to \phi D(\ker e_p)_{4k} \cap \phi(V^{4k}) \to \phi(V^{4k}) \xrightarrow{\rho} A \to 0,$$

where $A = \text{co-ker}\,(\phi D(\ker e_p) \cap \phi(V^{4k}) \to \phi(V^{4k}))$, ρ is the natural projection map, and $\phi D(V^{4k}) = \phi(V^{4k})$, by the previous lemma. Clearly, it suffices to compute $\dim_{Z_p} A$.

LEMMA 4.5. *The classes* $\rho\phi s(\lambda)$ $(\lambda \in \Pi'(p, k))$ *form a* Z_p*-basis of* A.

Proof. Suppose there exists a relation

$$\sum_{\lambda \in \Pi'(p,k)} a_\lambda \rho\phi s(\lambda) = 0;$$

then

$$\sum a_\lambda \phi s(\lambda) \in \ker \rho = \phi D(\ker e_p)_{4k} \cap \phi(V^{4k}).$$

Hence, by Proposition 2.4.1, $\sum a_\lambda \phi s(\lambda)$ is of the form $(P\phi(y))_{4k}$, for

$$y \in \sum_{i=0}^{k-1} H^{4i}(BSO; Z_p).$$

But, according to Lemma 4.3, the $\phi s(\lambda)$ are free over $\mathscr{A}/(\beta)$, so that the a_λ must be zero. Hence, the $\rho\phi s(\lambda)$ are free over Z_p.

Choose any $\omega \in \Pi(k)$ consisting entirely of multiples of $(p - 1)/2$. Then, according to Lemma 4.3,

$$\rho\phi s(\omega) = \sum a_{R',\lambda'} P^{R'} \rho\phi s(\lambda') = \sum a_{R',\lambda'} \rho(P^{R'} \phi s(\lambda')).$$

If R' is the zero sequence, then $P^{R'} = 1$. Otherwise,

$$P^{R'} \phi s(\lambda') \in \phi D(\ker e_p)_{4k} \cap \phi(V^{4k}) = \ker \rho,$$

by Proposition 2.4.1. Hence,

$$\rho\phi s(\omega) = \sum b_{\lambda'} \rho\phi s(\lambda'),$$

where λ' ranges over $\Pi'(p, k)$. Therefore, the $\rho(\phi s(\lambda'))$ generate A over Z_p. Q.E.D.

Theorem 5 now follows from the observation that

$$\dim_{Z_p} A = \text{cardinality } \Pi'(p, k) = \pi'(p, k).$$

Remarks. a) Clearly, it would have been sufficient to show that

$$\dim_{Z_p} A \geq \pi'(p, k),$$

and for this we need only have shown that the $\rho\phi s(\lambda')$ are linearly independent over Z_p.

b) It follows easily from Lemma 4.5 that

$$(\ker(e_p | V^*))_{4k} = \{(Py - y\,Wu\,P)_{4k} | \ y \in V^*\}.$$

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