A NOTE ON THE BORDISM ALGEBRA OF INVOLUTIONS

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1. INTRODUCTION

This note presents a structure theorem on the unoriented bordism group of involutions $\mathscr{R}_*(Z_2)$ defined in [1]. If we regard $\mathscr{R}_*(Z_2)$ as the bordism group of principal Z_2-bundles over closed manifolds, the tensor product of principal Z_2-bundles induces a multiplication in $\mathscr{R}_*(Z_2)$, making it an algebra over the Thom bordism algebra $\mathscr{R}_*$. On the other hand, we can also regard $\mathscr{R}_*(Z_2)$ as the singular bordism group $\mathscr{R}_*(B(Z_2))$ of a classifying space $B(Z_2)$. The diagonal map $\triangle: B(Z_2) \to B(Z_2) \times B(Z_2)$ then induces a comultiplication in $\mathscr{R}_*(Z_2)$, making it a coalgebra over $\mathscr{R}_*$. In summary, $\mathscr{R}_*(Z_2)$ becomes a Hopf algebra over $\mathscr{R}_*$. To study this Hopf algebra, we make use of the Smith homomorphism [1], whose existence is an additional special feature of $\mathscr{R}_*(Z_2)$. With all these structures on $\mathscr{R}_*(Z_2)$, we proceed to show that the Smith homomorphism helps to give some information on the comultiplication, which in turn yields some information on the multiplication. The information turns out to be just enough for a structure theorem. Our final conclusion states that $\mathscr{R}_*(Z_2)$ is an exterior algebra over $\mathscr{R}_*$ with generators in each dimension 2^n $(n = 0, 1, 2, \cdots)$. As we shall see, this theorem is quite formal in nature, and it supplements in a modest way the work of P. E. Conner and E. E. Floyd. There are quite a few places where we are unable to be more explicit. The author is grateful to Professor Conner for many useful conversations.

2. GENERALITIES

We recall here the definition of the singular bordism group $\mathscr{R}_*(X)$ of a space X. We consider pairs (M^n, f), where M^n is a closed n-manifold and $f: M^n \to X$ is a continuous map. Two such pairs (M_1^n, f_1) and (M_2^n, f_2) are *bordant* if there exists a compact $(n+1)$-manifold B^{n+1} and a map $F: B^{n+1} \to X$ such that the boundary of B^{n+1} is the disjoint union of M_1^n and M_2^n and $F \mid M_i^n = f_i$ $(i = 1, 2)$. This is an equivalence relation, and the equivalence class of (M, f) is denoted by $[M, f]$. (In [1], this class is denoted by $[M, f]_2$ to distinguish it from the oriented case.) The collection of all such classes is denoted by $\mathscr{R}_n(X)$, and $\mathscr{R}_*(X)$ is defined as $\Sigma_{n=0}^{\infty} \mathscr{R}_n(X)$. Disjoint union makes $\mathscr{R}_*(X)$ a vector space over Z_2. Moreover, $\mathscr{R}_*(X)$ is a module over the Thom unoriented bordism algebra $\mathscr{R}_*$. The module operation is given by $[M^n][N^m, f] = [M^n \times N^m, F]$, where F is the composition of projection onto N^m followed by f. The module $\mathscr{R}_*(X)$ is also augmented. The augmentation $\varepsilon: \mathscr{R}_*(X) \to \mathscr{R}_*$ is given by $\varepsilon[M^n, f] = [M^n]$. The augmentation kernel is denoted by $\tilde{\mathscr{R}}_*(X)$, and it is called the *reduced module*.

Corresponding to the diagonal map $\triangle: X \to X \times X$ we have the induced homomorphism

$$\triangle_*: \mathscr{R}_*(X) \to \mathscr{R}_*(X \times X).$$

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On the other hand, for any two spaces X and Y, we always have the natural homomorphism

$$\chi: \mathscr{R}_*(X) \otimes \mathscr{R}_*(Y) \to \mathscr{R}_*(X \times Y)$$

(here and in the sequel, tensor products are understood to be taken over $\mathscr{R}_*$), given by

$$\chi [M^n, f] \otimes [N^m, g] = [M^n \times N^m, f \times g].$$

In case X and Y are CW-complexes, as we shall henceforth assume, it is known [1, p. 75] that χ is an isomorphism. In particular, we may take $Y = X$ and consider $\chi^{-1} \circ \triangle_*$, which we also denote by $\triangle$.

PROPOSITION 2.1. *$\mathscr{R}_*(X)$ is a co-algebra over $\mathscr{R}_*$, with comultiplication $\triangle$ and co-unit ε. $\triangle$ is associative and commutative.*

For the definition of co-algebra we refer the reader to [2]. The proof of Proposition 2.1 is straightforward.

Suppose that X is an H-space. Let $m: X \times X \to X$ be the multiplication. We may then consider $m_* \circ \chi$, which we shall still denote by m. We also define $i: \mathscr{R}_* \to \mathscr{R}_*(X)$ by $i[M^n] = [M^n, \text{constant map}]$.

PROPOSITION 2.2. *$\mathscr{R}_*(X)$ is an algebra over $\mathscr{R}_*$ with multiplication* m *and unit* i. *The multiplication* m *is associative (commutative) if* X *is homotopy associative (commutative).*

Again the proof is easy. Although this is not needed in this note, we remark that Proposition 2.2 remains valid if one uses a weaker notion of cohomology H-space, defined in an obvious way, instead of H-space.

The following assertion can be verified without difficulty.

PROPOSITION 2.3. *If* X *is an* H*-space, then $\mathscr{R}_*(X)$ is a Hopf algebra over $\mathscr{R}_*$, with algebra $(\mathscr{R}_*(X), m, i)$ and co-algebra $(\mathscr{R}_*(X), \triangle, \varepsilon)$.*

3. THE STRUCTURE OF $\mathscr{R}_*(Z_2)$

We now specialize to the case where $X = B(Z_2)$ is a classifying space for the group Z_2. We take $B(Z_2)$ to be the infinite-dimensional real projective space, so that $B(Z_2)$ is a CW-complex and Proposition 2.1 applies. Let $c \in H'(B(Z_2); Z_2)$ be the fundamental class; then there exists a map $m: B(Z_2) \times B(Z_2) \to B(Z_2)$, unique up to homotopy, such that

$$m^*(c) = c \otimes 1 + 1 \otimes c \in H'(B(Z_2) \times B(Z_2); Z_2).$$

This defines an associative and commutative H-space structure on $B(Z_2)$. According to Proposition 2.3, $\mathscr{R}_*(B(Z_2))$ is a Hopf algebra. It is often convenient to interpret $\mathscr{R}_*(B(Z_2))$ as the bordism group of fixed-point-free involutions on manifolds. We consider pairs (M^n, T), where M^n is a closed n-manifold and T a differentiable fixed-point-free involution on M^n. Then there exists a map $f: M^n/T \to B(Z_2)$, unique up to homotopy, associated with (M^n, T). We say that (M_1^n, T_1) and (M_2^n, T_2) are *bordant* if the corresponding bordism classes $[M_1^n/T_1, f_1]$ and $[M_2^n/T_2, f_2]$ are the same. This means there is a compact $(n+1)$-manifold B^{n+1}

and a differentiable fixed-point-free involution T on B^{n+1} such that $\partial B^{n+1} = M_1^n \cup M_2^n$ and $T \mid M_i^n = T_i$ (i = 1, 2). In terms of involution, the multiplication of $[M_1^n, T_1]$ and $[M_2^m, T_2]$ is merely the tensor product of Z_2-bundles $M_1^n \to M_1^n/T_1$ and $M_2^m \to M_2^m/T_2$. Explicitly, one considers involutions $T_1 \times T_2$, $T_1 \times 1$, and $1 \times T_2$ on $M_1^n \times M_2^m$; then both $T_1 \times 1$ and $1 \times T_2$ induce the same involution T on $M_1^n \times M_2^n/T_1 \times T_2$, which is differentiable and fixed-point-free. We have then the relation

$$[M_1^n, T_1][M_2^m, T_2] = [M_1^n \times M_2^m/T_1 \times T_2, T].$$

As in [1], we shall denote $\mathscr{R}_*(B(Z_2))$ by $\mathscr{R}_*(Z_2)$.

Let us summarize here what is known about $\mathscr{R}_*(Z_2)$. It has been shown in [1, Theorem 23.2; p. 60] that $\mathscr{R}_*(Z_2)$ is a free module over $\mathscr{R}_*$. A homogeneous $\mathscr{R}_*$-basis can be taken as $[S^n, A]$ (n = 0, 1, 2, ⋯), where S^n is the n-sphere and A the antipodal involution on S^n. Then we have the Smith homomorphism

$$S: \mathscr{R}_*(Z_2) \to \mathscr{R}_*(Z_2).$$

(In [1] it is denoted by $\triangle$. Here we reserve $\triangle$ for the comultiplication.) This is an $\mathscr{R}_*$-homomorphism of degree -1, and it can be described as follows. Let $[M^n, f]$ be an element of $\mathscr{R}_n(Z_2)$; take an (n - 1)-submanifold $N^{n-1} \subset M^n$, dual to the class $f^*(c) \in H^1(M^n; Z_2)$; and set $g = f \mid N^{n-1}$. Then

$$S[M^n, f] = [N^{n-1}, g] \in \mathscr{R}_{n-1}(Z_2).$$

Geometrically, we may assume that $f(M^n) \subset P^k \subset B(Z_2)$, where P^k is the real projective k-space. We may also assume that $f: M^n \to P^k$ is differentiable and transverse-regular [1; p. 21] on $P^{k-1} \subset P^k$. Under these conditions, we can take $N^{n-1} = f^{-1}(P^{k-1})$. From this it is clear that $S[S^n, A] = [S^{n-1}, A]$ for all n (with the understanding that $[S^{-1}, A] = 0$). In particular, Ker S = $i(\mathscr{R}_*)$. Thus the image under S of the basis $[S^n, A]$ is fairly simple. However, the basis is not quite convenient for algebraic manipulation, because $[S^n, A]$ is not in $\tilde{\mathscr{R}}_*(Z_2)$, for even n. The following proposition is a technical device to take care of this.

PROPOSITION 3.1. *There exists a unique* $\mathscr{R}_*$*-basis* $\{x_n\}_{n=0}^{\infty}$ *in* $\mathscr{R}_*(Z_2)$ *with the following properties.*

(i) $x_0 = [S^0, A]$,

(ii) $x_n \in \tilde{\mathscr{R}}_n(Z_2)$ *for all* $n \geq 1$,

(iii) $S(x_n) = x_{n-1}$ *for all* $n \geq 1$.

Proof. One can show by induction that if such a basis exists, then

$$x_n = [S^n, A] + \sum_{j=0}^{n-1} [P^{n-j}]x_j.$$

Conversely, this relation determines inductively a sequence $\{x_n\}_{n=0}^{\infty}$ that forms an $\mathscr{R}_*$-basis with all the properties (i), (ii), and (iii).

We now come to our first important observation.

PROPOSITION 3.2. *For each* $x \in \mathscr{R}_*(Z_2)$, *we have the relation*

$$\Delta S(x) = (S \otimes 1)\Delta(x).$$

Proof. Define the mapping

$$S_1: \mathscr{R}_*(B(Z_2) \times B(Z_2)) \to \mathscr{R}_*(B(Z_2) \times (B(Z_2))$$

as follows. If $[M^n, f] \in \mathscr{R}_*(B(Z_2) \times B(Z_2))$, we may assume that $f(M^n) \subset P^k \times P^k$ for some large k and that $f: M^n \to P^k \times P^k$ is differentiable and transverse regular on $P^{k-1} \times P^k \subset P^k \times P^k$. Let

$$N^{n-1} = f^{-1}(P^{k-1} \times P^k) \quad \text{and} \quad g = f \mid N^{n-1}.$$

Then define $S_1[M^n, f] = [N^{n-1}, g]$. It is not difficult to verify that S_1 is a well-defined $\mathscr{R}_*$-homomorphism of degree -1. Now consider the diagram

$$\begin{array}{ccccc} \mathscr{R}_*(Z_2) & \xrightarrow{\Delta_*} & \mathscr{R}_*(B(Z_2) \times B(Z_2)) & \xleftarrow{\chi} & \mathscr{R}_*(Z_2) \otimes \mathscr{R}_*(Z_2) \\ \downarrow S & & \downarrow S_1 & & \downarrow S \otimes 1 \\ \mathscr{R}_*(Z_2) & \xrightarrow{\Delta_*} & \mathscr{R}_*(B(Z_2) \times B(Z_2)) & \xleftarrow{\chi} & \mathscr{R}_*(Z_2) \otimes \mathscr{R}_*(Z_2) . \end{array}$$

To see that it is commutative, one needs only observe that $\Delta: P^k \to P^k \times P^k$ is transverse regular on $P^{k-1} \times P^k$, with $\Delta^{-1}(P^{k-1} \times P^k) = P^{k-1}$. This proves Proposition 3.2.

PROPOSITION 3.3. *The comultiplication in* $\mathscr{R}_*(Z_2)$ *is given by the formula*

$$\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}.$$

Proof. The assertion is trivial for $n = 0$. Suppose it is true for $k \leq n$. Then

$$(S \otimes 1)\Delta(x_{n+1}) = \Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i} = (S \otimes 1) \sum_{i=1}^{n} x_i \otimes x_{n+1-i}.$$

Since $\operatorname{Ker}(S \otimes 1) = i(\mathscr{R}_*) \otimes \mathscr{R}_*(Z_2)$, it follows that

$$\Delta(x_{n+1}) = \sum_{i=1}^{n+1} x_i \otimes x_{n+1-i} + x_0 \otimes x,$$

where $x \in \mathscr{R}_{n+1}(Z_2)$. But $x = x_{n+1}$, because ε is a co-unit. Hence

$$\Delta(x_{n+1}) = \sum_{i=0}^{n+1} x_i \otimes x_{n+1-i}.$$

We proceed to study the multiplication in $\mathscr{R}_*(Z_2)$.

PROPOSITION 3.4. *If* $x \in \tilde{\mathscr{R}}_*(Z_2)$, *then* $x^2 = 0$.

Proof. It suffices to show that $[S^n, A]^2 \equiv 0 \bmod \mathscr{R}_*$. Let $d \in H^1(P^n; Z_2)$ be the generator; then the characteristic class of $[S^n, A]^2$ is

$$d \otimes 1 + 1 \otimes d \in H^1(P^n \times P^n; Z_2).$$

Let $W = \Sigma_{i=0}^{2n} W_i$ be the Whitney class of $P^n \times P^n$. For any partition $(k_1, k_2, \cdots, k_p)$ of 2n, consider the involution number [1; p. 60]

$$\langle (d \otimes 1 + 1 \otimes d)^{k_1} W_{k_2} \cdots W_{k_p}, P^n \times P^n \rangle. \tag{1}$$

According to [1, Theorem 23.1, p. 60], our assertion is true if all the involution numbers of the form (1) with $k_1 > 0$ vanish. If n is even, then $d \otimes 1 + 1 \otimes d = W_1$. Hence (1) reduces to a Whitney number of $P^n \times P^n$ involving W_1. But these numbers vanish, because $P^n \times P^n$ is bordant to the complex projective n-space, which is orientable. If n is odd and k_1 is odd, then k_i is odd for some i. In this case (1) vanishes because $W_{k_i} = 0$. Finally, if n is odd and all k_i are even, then (1) is a linear combination of products of the form

$$\langle f(d^2), P^n \rangle \langle g(d^2), P^n \rangle,$$

where f and g are polynomials. But obviously they all vanish.

There is also a geometrical proof of Proposition 3.4, suggested by P. E. Conner. We outline it briefly. Take $[M^n, T] \in \tilde{\mathscr{R}}_n(Z_2)$. Let M^n be imbedded in $M^n \times M^n$ by the diagonal map. Let σ be the diagonal involution $\sigma(x, y) = (Tx, Ty)$ on $M^n \times M^n$, and $\tilde{T}$ the involution on $M^n \times M^n/\sigma$ induced by $T \times 1$. Then

$$[M^n, T]^2 = [M^n \times M^n/\sigma, \tilde{T}].$$

It suffices to show that $S[M^n \times M^n/\sigma, \tilde{T}] = 0$, since S is a monomorphism on $\tilde{\mathscr{R}}_*(Z_2)$. Choose a closed tubular neighborhood [1; p. 58] N of M^n, invariant under σ. Let

$$B_1 = M^n \times M^n - \text{Int } N/\sigma \quad \text{and} \quad B_2 = N/\sigma;$$

then B_1 and B_2 are compact 2n-manifolds in $M^n \times M^n/\sigma$ with $B_1 \cap B_2 = \partial N/\sigma$. Since T is free on M^n, we can take N small enough so that $(x, y) \in \text{Int } N$ implies $(Tx, y) \notin N$. This means that $B_2 = \tilde{T}(B_1)$. According to [1, Theorem 26.2, p. 68], we have the relation

$$S[M^n \times M^n/\sigma, \tilde{T}] = [\partial N/\sigma, \tilde{T}].$$

Now $\partial N/\sigma$ is precisely the tangent sphere bundle of M^n/T, and $\tilde{T}$ is the antipodal involution on each fiber. Since $[M^n/T] = 0$, it is obvious that $[\partial N/\sigma, \tilde{T}] = 0$.

Next we try to obtain information on multiplication of $\mathscr{R}_*(Z_2)$ in general. Let $(i, j) = \binom{i+j}{j} = (i+j)!/i!\,j!$ be the (mod 2)-combinatorial coefficient, where we agree that $(i, j) = 0$ if either $i < 0$ or $j < 0$. For each $n \geq 0$, let $A_n \subset \mathscr{R}_*(Z_2)$ be the $\mathscr{R}_*$-module generated by $x_0, x_1, \cdots, x_{n-1}$, and let

$$B_n = \sum_{i+j \leq n} A_i \otimes A_j \subset \mathscr{R}_*(Z_2) \otimes \mathscr{R}_*(Z_2).$$

Notice that by Proposition 3.3, for each $x \in \mathscr{R}_*(Z_2)$, we have the relation $x \in A_n$ if and only if $\tilde{\Delta}(x) \in B_n$, where $\tilde{\Delta}$ is given by $\tilde{\Delta}(x) = \Delta x - (x \otimes x_0 + x_0 \otimes x)$.

PROPOSITION 3.5. *The multiplication in* $\mathscr{R}_*(Z_2)$ *satisfies*

$$x_n x_m \equiv (n, m) x_{n+m} \quad \text{mod } A_{n+m}.$$

Proof. The assertion is trivial if $n + m = 0$. Suppose $n + m = \ell$ and the assertion is true for $n + m < \ell$. Then

$$\Delta(x_n x_m) = \sum_{i=0}^{n} \sum_{j=0}^{m} x_i x_j \otimes x_{n-i} x_{m-j}$$

$$\equiv x_n x_m \otimes x_0 + x_0 \otimes x_n x_m + \sum_{ij} (i, j)(n-i, m-j) x_{i+j} \otimes x_{\ell-i-j} \quad \text{mod } B_\ell,$$

that is,

$$\tilde{\Delta}(x_n x_m) \equiv \sum_{0<k<\ell} \left[\sum_{i=0}^{n} (i, k-i)(n-i, m-k+i) \right] x_k \otimes x_{\ell-k} \quad \text{mod } B_\ell.$$

Comparing the coefficients of t^n in the identity $(1+t)^k (1+t)^{\ell-k} = (1+t)^\ell$, we obtain the identity

$$\sum_{i=0}^{n} (i, k-i)(n-i, m-k+i) = (n, m),$$

and it follows that

$$\tilde{\Delta}(x_n x_m - (n, m) x_{n+m}) \equiv 0 \quad \text{mod } B_{n+m}.$$

By the remark just made,

$$x_n x_m \equiv (n, m) x_{n+m} \quad \text{mod } A_{n+m}.$$

The congruence cannot be replaced by an actual equation. For example, one can compute explicitly that

$$x_1 x_2 = x_3 + [P^2] x_1.$$

We do not have a general formula to account for the residue terms involved. But fortunately, we are still able to describe the algebra $\mathscr{R}_*(Z_2)$.

THEOREM. $\mathscr{R}_*(Z_2)$ *is the exterior algebra over* $\mathscr{R}_*$ *generated by* $\{x_{2^n}\}_{n=0}^{\infty}$.

Proof. For each integer $n \geq 1$, let $\omega(n) = (k_1, k_2, \cdots, k_p)$ $(k_1 > k_2 > \cdots > k_p \geq 0)$ be the diadic expansion of n; that is, let

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}.$$

Let $x_{\omega(n)} = x_{2^{k_1}} x_{2^{k_2}} \cdots x_{2^{k_p}}$. We have to show that the $x_{\omega(n)}$ $(n = 1, 2, \cdots)$ together with x_0 form an $\mathscr{R}_*$-basis. Recall that $(n, m) \neq 0$ if and only if the diadic expansions of n and m have no common part. It follows from Proposition 3.5 that

$$x_{\omega(n)} \equiv x_n \quad \mod A_n.$$

This means that the $x_{\omega(n)}$ $(n = 1, 2, \cdots)$ are related to the x_n $(n = 1, 2, \cdots)$ by a triangular matrix with identity on the diagonal. Therefore our description of the algebra $\mathscr{R}_*(Z_2)$ is correct.

To have a complete description of the Hopf algebra $\mathscr{R}_*(Z_2)$, we ought to express the comultiplication in terms of the basis $x_{\omega(n)}$. From Proposition 3.5, we deduce easily that

$$\Delta x_{\omega(n)} \equiv \sum_{i=0}^{n} x_{\omega(i)} \otimes x_{\omega(n-i)} \quad \mod B_n.$$

But again we do not know the exact formula for the residue terms. Direct calculation shows that for $n \leq 7$ there is no residue term. But

$$\Delta x_8 = \sum_{i=0}^{8} x_{\omega(i)} \otimes x_{\omega(n-i)} + [P^2](x_4 \otimes x_2 + x_2 \otimes x_4) + [P^2]^2(x_2 x_1 \otimes x_1 + x_1 \otimes x_1 x_2).$$

Another relevant problem is the relation between S and the multiplication. specifically, is S a derivation? Once more, all we can say is that

$$S(x_n x_m) \equiv S(x_n) x_m + x_n S(x_m) \quad \mod A_{n+m-1}$$

with nonvanishing residue terms in general.

REFERENCES

1. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik, Vol. 33, Springer Verlag, Berlin-Göttingen-Heidelberg, (1964).

2. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, mimeographed notes, Princeton.

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