RATIONAL APPROXIMATION TO $|x|$

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The function $|x|$ plays a central role in the theory of polynomial approximation. Indeed, Lebesgue's proof of the Weierstrass approximation theorem is based solely on the fact that the single function $|x|$ can be approximated. One can even give a proof of *Jackson's* approximation theorem by simply using an appropriate polynomial approximation to $|x|$.

Quantitatively speaking, $|x|$ has "order of approximation" $1/n$. In precise language this means that (with C_1, C_2 positive absolute constants)

(1) there exists an n^{th}-degree polynomial $P(x)$ such that, throughout $[-1, 1]$,

$$\big| |x| - P(x) \big| \leq \frac{C_1}{n};$$

(2) there does not exist an n^{th}-degree polynomial $P(x)$ such that, throughout $[-1, 1]$,

$$\big| |x| - P(x) \big| \leq \frac{C_2}{n}.$$

Suppose we now turn to the question of approximation by rational functions rather than by polynomials. While it is true that in this context the function $|x|$ loses much of its special significance, it is nevertheless of some interest to determine the order of approximation to $|x|$ by n^{th}-order rational functions (the problem was actually suggested by H. S. Shapiro).

Now it is known that, in some overall sense, rational approximation is essentially no better than polynomial approximation, and this suggests the naïve guess that the order of approximation of $|x|$ by n^{th}-order rational functions is also $1/n$. The truth, however, is remarkably far from this guess. Indeed, the purpose of the present paper is to show that this order of approximation is actually $e^{-c\sqrt{n}}$.

Notation. n is an integer greater than 4, $\xi = \exp(-n^{-1/2})$, and

$$p(x) = \prod_{k=0}^{n-1} (x + \xi^k), \qquad r(x) = x\,\frac{p(x) - p(-x)}{p(x) + p(-x)}.$$

By the *order* of a rational function we mean the maximum of the degrees of its numerator and denominator. [Note that the order of $r(x)$ is n when n is even].

THEOREM (A). $\big| |x| - r(x) \big| \leq 3e^{-\sqrt{n}}$ *throughout* $[-1, 1]$.

(B). *There does not exist an* n^{th}*-order rational function* $R(x)$ *such that* $\big| |x| - R(x) \big| \leq \frac{1}{2} e^{-9\sqrt{n}}$ *throughout* $[-1, 1]$.

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Remark. The gap between the exponents $-\sqrt{n}$ and $-9\sqrt{n}$ can be narrowed somewhat without much change in the proof; but we are far from eliminating it. Only in this crude sense then can we justify the statement, made in the introduction, that the order of approximation is $e^{-c\sqrt{n}}$.

LEMMA 1. $\prod_{j=1}^{n} \frac{1-\xi^j}{1+\xi^j} \leq e^{-\sqrt{n}}$.

Proof. Since $\frac{1-t}{1+t} \leq e^{-2t}$ for all $t \geq 0$, it follows that

$$\prod_{j=1}^{n} \frac{1-\xi^j}{1+\xi^j} \leq \exp\left(-2\sum_{j=1}^{n} \xi^j\right) = \exp\left(-2\xi \frac{1-\xi^n}{1-\xi}\right). \tag{3}$$

A simple computation shows that, since $n > 4$,

$$2\xi(1-\xi^n) \geq 1. \tag{4}$$

We also recall that $1 - t \leq e^{-t}$, and we conclude that

$$\frac{1}{1-\xi} = \frac{1}{1-\exp(-n^{-1/2})} \geq \sqrt{n}. \tag{5}$$

Inequalities (3), (4), and (5) now combine to give the lemma.

LEMMA 2. *For* $\exp(-\sqrt{n}) \leq x \leq 1$, $\left|\frac{p(-x)}{p(x)}\right| \leq \exp(-\sqrt{n})$.

Proof. Suppose that $\xi^{j+1} \leq x \leq \xi^j$, $0 \leq j \leq n$. Then

$$\left|\frac{p(-x)}{p(x)}\right| = \prod_{k=0}^{j} \frac{\xi^k - x}{\xi^k + x} \prod_{k=j+1}^{n-1} \frac{x-\xi^k}{x+\xi^k} \leq \prod_{k=0}^{j} \frac{\xi^k - \xi^n}{\xi^k + \xi^n} \prod_{k=j+1}^{n-1} \frac{\xi^j - \xi^k}{\xi^j + \xi^k}$$

$$= \prod_{m=n-j}^{n} \frac{1-\xi^m}{1+\xi^m} \prod_{m=1}^{n-j-1} \frac{1-\xi^m}{1+\xi^m} = \prod_{m=1}^{n} \frac{1-\xi^m}{1+\xi^m},$$

and the result follows from Lemma 1.

Proof of (A). Since $|x|$ and $r(x)$ are both even, it suffices to prove the required inequality for $[0, 1]$. For $0 \leq x \leq \exp(-\sqrt{n})$ this is quite trivial, since here $p(-x) \geq 0$, so that $0 \leq r(x) \leq x$. Hence

$$\big|\,|x| - r(x)\big| = x - r(x) \leq x \leq \exp(-\sqrt{n}).$$

For $\exp(-\sqrt{n}) \leq x \leq 1$, on the other hand, we see that

$$|x - r(x)| = 2x\left|\frac{p(-x)}{p(x)+p(-x)}\right| = \frac{2x}{\left|1 + \frac{p(x)}{p(-x)}\right|} \leq \frac{2}{\left|\frac{p(x)}{p(-x)}\right| - 1}. \tag{6}$$

Applying Lemma 2, we conclude from (6) that $|x - r(x)| \leq 2/(\exp\sqrt{n} - 1)$, and since $n > 2$, $2/(\exp\sqrt{n} - 1) \leq 3\exp(-\sqrt{n})$. The proof is complete.

LEMMA 3. *Let* $b \geq a \geq 0$, *and let* ξ *be any complex number. Then*

$$\int_a^b \log\left|\frac{t+\xi}{t-\xi}\right| \frac{dt}{t} \geq \frac{-\pi^2}{2}.$$

Proof. Write $\xi = u + iv$ (u, v real). We may clearly assume that $u \neq 0$. Then, for $t \geq 0$,

$$\left|\frac{t+\xi}{t-\xi}\right| = \left(\frac{(t+u)^2 + v^2}{(t-u)^2 + v^2}\right)^{1/2} \geq \left(\frac{(t-|u|)^2}{(t+|u|)^2}\right)^{1/2} = \left|\frac{t-|u|}{t+|u|}\right|.$$

Hence

$$\int_a^b \log\left|\frac{t-\xi}{t+\xi}\right| \frac{dt}{t} \geq \int_a^b \log\left|\frac{t-|u|}{t+|u|}\right| \frac{dt}{t} = \int_{a/|u|}^{b/|u|} \log\left|\frac{t-1}{t+1}\right| \frac{dt}{t}$$

$$\geq \int_0^\infty \left|\frac{t-1}{t+1}\right| \frac{dt}{t} = \frac{-\pi^2}{2}.$$

LEMMA 4. *Let* $P(x)$, *not identically* 0, *be any* n^{th}*-degree polynomial. Then there exists a point in* $[e^{-\sqrt{n}}, 1]$ *where* $\left|x\frac{P(-x)}{P(x)}\right| > \exp(-6\sqrt{n})$.

Proof. Write $\delta = \exp(-\sqrt{n})$. If the lemma were false, we would have the relation

$$\int_\delta^1 \log\left|t\frac{P(-t)}{P(t)}\right| \frac{dt}{t} \leq -6\sqrt{n}\int_\delta^1 \frac{dt}{t} = -6n. \tag{7}$$

On the other hand, we have the equation

$$\int_\delta^1 \log\left|t\frac{P(-t)}{P(t)}\right| \frac{dt}{t} = \int_\delta^1 \frac{\log t}{t}\, dt + \sum_\xi \int_\delta^1 \log\left|\frac{t+\xi}{t-\xi}\right| \frac{dt}{t}, \tag{8}$$

where ξ runs through the zeros of $P(t)$. Noting that $\int_\delta^1 t^{-1}\log t\, dt = -n/2$ and applying Lemma 3 to the n terms in the right member of (8), we obtain

$$\int_\delta^1 \log\left|t\frac{P(-t)}{P(t)}\right| \frac{dt}{t} \geq -\frac{n}{2}(1+\pi^2). \tag{9}$$

The contradiction between (7) and (9) gives the lemma.

Proof of (B). Assume that there exists an $R(x)$ satisfying the inequality. Choosing

$$R_1(x) = \frac{R(x) + R(-x)}{2} - R(0),$$

we note that $R_1(x)$ is even and of order $2n$, and vanishes at the origin. Also we see that

$$(10) \qquad \big|\,|x| - R_1(x)\big| \leq \exp(-9\sqrt{n})$$

throughout $[-1, 1]$.

Now write $R_1(x) = x^2 S(x^2)/Q(x^2)$, where $Q(x^2) > 0$ throughout $[-1, 1]$, and observe that, by (10), $S(x^2) > 0$ for $\exp(-\sqrt{2n}) \leq x \leq 1$. Hence, for $\exp(-\sqrt{2n}) \leq x \leq 1$,

$$(11) \qquad |x - R_1(x)| = \left| x \frac{Q(x^2) - xS(x^2)}{Q(x^2)} \right| \geq \left| x \frac{Q(x^2) - xS(x^2)}{Q(x^2) + xS(x^2)} \right|.$$

The application of Lemma 4 to the polynomial $P(x) = Q(x^2) + xS(x^2)$ insures the existence, in $\exp(-\sqrt{2n}) \leq x \leq 1$, of a point where

$$(12) \qquad \left| x \frac{Q(x^2) - xS(x^2)}{Q(x^2) + xS(x^2)} \right| > \exp(-6\sqrt{2n}) > \exp(-9\sqrt{n}).$$

Together, (11) and (12) contradict (10), and the proof is complete.

We conclude with the following observation: Approximation of $|x|$ in $[-1, 1]$ is equivalent to approximation of $\sqrt{x}$ in $[0, 1]$. In fact, if $R(x)$ approximates to $\sqrt{x}$, then $R(x^2)$ approximates to $|x|$; and conversely, if $R(x)$ approximates to $|x|$, then $\frac{R(\sqrt{x}) + R(-\sqrt{x})}{2}$ approximates to $\sqrt{x}$. Thus we conclude that the order of approximation to $\sqrt{x}$ is also $\exp(-c\sqrt{n})$. This degree of approximation is also possible for x^α, where α is any positive noninteger rational.

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