CONCERNING THE ORDER STRUCTURE OF KÖTHE SEQUENCE SPACES

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1. INTRODUCTION

Since 1934 a rather extensive study has been made, principally by G. Köthe, of the linear and topological structure of certain vector spaces λ of real sequences for which there is associated a dual sequence space λ^*. An excellent account of the basic theory of these spaces can be found in Section 30 of [4]. In 1951, this theory was extended to spaces of functions by J. Dieudonné [2]. These spaces are partially ordered in a natural way, and the purpose of this paper is to study some of the relations that exist between their order and topological structures. More specifically, we shall concern ourselves with some of the properties of the coarsest and finest compatible topologies (see Section 3) on these spaces. Since some of our results hold only for the sequence space case, we shall place our presentation in this framework and then indicate what modifications must be made for function spaces.

2. PRELIMINARY MATERIAL

Suppose ω denotes the ordered vector space of all sequences $x = (x_i)$ of real numbers and that ϕ denotes the linear subspace of ω consisting of those sequences with only finitely many non-zero components. (The linear operations and order relations on ω are defined in the usual coordinatewise fashion; for example, $x \geq y$ if $x_i \geq y_i$ for all i.) If λ is a linear subspace of ω containing ϕ, the *α-dual* λ^* of λ is defined by

$$\lambda^* = \{u = (u_i) \in \omega\colon \sum |x_i u_i| < +\infty \text{ for all } x = (x_i) \in \lambda\}.$$

(In this paper, Σ will indicate summation over the index set of all natural numbers.) The spaces λ and λ^* form a dual system $<\lambda, \lambda^*>$ with respect to the bilinear form $(x, u) \to <x, u> = \Sigma x_i u_i$. If $K = \{x \in \lambda\colon x_i \geq 0 \text{ for all } i\}$ denotes the cone in λ, the dual cone $K' = \{u \in \lambda^*\colon <x, u> \geq 0 \text{ for all } x \in K\}$ coincides with the cone of all sequences in λ^* with non-negative components.

The α-dual of λ^*, which we shall denote by λ^{**}, contains λ; if $\lambda = \lambda^{**}$, then λ is said to be *perfect.* For a given $s = (s_i)$ define

$$\lambda_s = \{u \in \omega\colon \sum |s_i u_i| < +\infty\};$$

then $\lambda = \bigcap_{v \in K'} \lambda_v$ if λ is perfect. Moreover,

$$\lambda = \bigcup_{y \in K} \lambda_y^*$$

if λ is solid (that is, if $|x| \leq |y|$ and $y \in \lambda$ imply $x \in \lambda$, where $|x| = (|x_i|)$ denotes the lattice theoretic absolute value of x in λ).

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Two sequence spaces λ and μ are *isomorphic* if there exist one-one, onto, linear maps $\sigma: \lambda \to \mu$, $\rho: \lambda^* \to \mu^*$ such that $<x, u> = <\sigma(x), \rho(u)>$ for all $x \in \lambda$, $u \in \lambda^*$. If λ and μ are isomorphic and the mappings σ, ρ are order isomorphisms, we say that λ and μ are *order isomorphic*. If λ and μ are isomorphic (respectively, order isomorphic), they are topologically isomorphic for any pair of corresponding topologies that can be defined entirely in terms of the dual system (respectively, the dual system and order structure). If $u \geq 0$, the following alternatives hold for λ_u, λ_u^* (see [4, p. 414]):

(1) λ_u is order isomorphic to ω and λ_u^* is order isomorphic to ϕ if $u_i = 0$ for all but finitely many values of i.

(2) λ_u is order isomorphic to the space

$$\ell^1 = \{x \in \omega: \sum |x_i| < +\infty\}$$

and λ_u^* is order isomorphic to the space (m) of bounded sequences if $u_i \neq 0$ for all but finitely many values of i.

(3) λ_u is order isomorphic to $\omega \times \ell^1$, and λ_u^* is order isomorphic to $\phi \times (m)$ otherwise.

Köthe defines the *normal topology* on a sequence space λ to be the locally convex topology generated by the family of semi-norms

$$p_u(x) = \sum |x_i u_i| \qquad (u \in K').$$

A simple computation shows that this topology coincides with the topology $o(\lambda, \lambda^*)$ on λ of uniform convergence on the order bounded sets in λ^*, which was studied for more general dual systems in [6]. (A subset B of λ^* is order bounded if it is contained in an order interval $[u, v] = \{s \in \lambda^*: u \leq s \leq v\}$.) Thus a neighborhood basis of the zero element 0 for the normal topology is given by the class

$$\{[-u, u]^\circ: u \in K'\}$$

where $^\circ$ denotes the formation of the polar set for the dual system $<\lambda, \lambda^*>$.

We refer the reader to [5] or [8] for the basic results on ordered vector spaces; [3] contains a discussion of some of the fundamental properties of the order structure of the spaces introduced in [2].

3. THE EXTREME COMPATIBLE TOPOLOGIES

Throughout this section, we assume that λ is a sublattice of ω and that $\lambda \supset \phi$. A Hausdorff locally convex topology $\mathfrak{T}$ on λ which is finer than $\sigma(\lambda, \lambda^*)$ is a *compatible topology* if the positive cone K in λ is $\mathfrak{T}$-normal (see [8; Def. 1, p. 121]) and the lattice operations are $\mathfrak{T}$-continuous. In this paper, we discuss some of the properties of the finest and the coarsest compatible topologies on λ.

The coarsest compatible topology on λ is the topology $o(\lambda, \lambda^*)$ of uniform convergence on the order bounded subsets of λ^*. This is a consequence of Proposition 1.1 and Theorem 2.1 of [6]. We shall first characterize those spaces λ for which the coarsest compatible topology coincides with the weak topology $\sigma(\lambda, \lambda^*)$.

PROPOSITION 1. *The lattice operations in* λ *are* $\sigma(\lambda, \lambda^*)$-*continuous if and only if* $\lambda^* = \phi$.

Proof. Suppose that the lattice operations in λ are $\sigma(\lambda, \lambda^*)$-continuous; then, since K is always $\sigma(\lambda, \lambda^*)$-normal, it must be that $o(\lambda, \lambda^*) = \sigma(\lambda, \lambda^*)$. It follows that each order interval in λ^* is contained in the closed convex circle hull of some finite set in λ^*; hence each order interval in λ^* is contained in a finite dimensional subspace of λ^*.

Since $\lambda^* \supset \phi$ in general, it is only necessary to show that $\lambda^* \subset \phi$. Suppose, to the contrary, that there exists a $u \in \lambda^*$ such that $u_{i_k} \neq 0$ $(k = 1, 2, 3, \cdots)$. Without loss in generality, we can assume that $u_i \geq 0$ for all i. For each positive integer k, define $v^{(k)}$ to be the i_k-section of u (that is, the i_k-th Abschitt in Köthe's terminology (see [4; p. 415]). Then one can easily verify that $v^{(k)} \in [-u, u]$ for each k, and that $\{v^{(k)}: k = 1, 2, \cdots\}$ is a linearly independent set. But then the order interval $[-u, u]$ cannot be contained in a finite dimensional subspace of λ^*; hence $\lambda^* = \phi$.

Conversely, suppose that $\lambda^* = \phi$ and that $[-v, v]$ $(v \in K')$ is a given order interval in λ^*. Suppose that v_{i_k} $(k = 1, 2, \cdots, m)$ denote the non-zero components of v, then define $v^{(k)}$ to be the sequence with v_{i_k} as its i_k-component and all other components equal to zero. Then if $u \in [-v, v]$, we can write

$$u = \sum_{k=1}^{m} (u_{i_k}/mv_{i_k})mv^{(k)} \quad \text{with} \quad \sum_{k=1}^{m} |u_{i_k}/mv_{i_k}| \leq 1 .$$

It follows that $[-v, v]$ is contained in the convex circled hull of the finite set $F = \{mv^{(k)}: k = 1, 2, \cdots, m\}$. Therefore, $\sigma(\lambda, \phi) = o(\lambda, \phi)$ which implies that the lattice operations in λ are $\sigma(\lambda, \phi)$-continuous.

COROLLARY. *If λ is perfect, the lattice operations in λ are $\sigma(\lambda, \lambda^*)$ continuous if and only if $\lambda = \omega$.*

The preceding proposition shows that the lattice operations in λ are rarely weakly continuous. In contrast to this result, we shall now prove that these operations are always sequentially continuous for the weak topology.

PROPOSITION 2. *The lattice operations in λ are $\sigma(\lambda, \lambda^*)$-sequentially continuous.*

Proof. Suppose that $x^{(n)}$ is a sequence in λ that converges for $\sigma(\lambda, \lambda^*)$ to $x^{(o)} \in \lambda$; then this sequence converges to the same limit in λ^{**} for $\sigma(\lambda^{**}, \lambda^*)$. But λ^{**} is perfect; hence λ^{**}, equipped with $\sigma(\lambda^{**}, \lambda^*)$, is the projective limit of the spaces $\lambda_u (u \in K')$, each equipped with $\sigma(\lambda_u, \lambda_u^*)$ (see [4; Section 30, p. 416]). Therefore, $x^{(n)}$ converges to $x^{(o)}$ in each λ_u for $\sigma(\lambda_u, \lambda_u^*)$. It is an immediate consequence from Section 2 that each λ_u, equipped with $\sigma(\lambda_u, \lambda_u^*)$, is order isomorphic to the topological product of ℓ^1, equipped with $\sigma(\ell^1, (m))$, and ω, equipped with $\sigma(\omega, \phi)$. Hence, in virtue of Proposition 1 and the fact that $\sigma(\ell^1, (m))$-convergence for sequences in equivalent to convergence in the usual ℓ^1-norm, it follows that $x^{(n)}$ converges to $x^{(o)}$ in each λ_u for $\sigma(\lambda_u, \lambda_u^*)$. By making use of known results concerning the projective limit (see [4; Section 19.10, p. 234]), we conclude that $x^{(n)}$ converges to $x^{(o)}$ for $\sigma(\lambda^{**}, \lambda^*)$ and hence for $\sigma(\lambda, \lambda^*)$.

COROLLARY. *The topology $\sigma(\lambda, \lambda^*)$ is metrizable if and only if $\lambda^* = \phi$.*

Proof. The necessity is an immediate consequence of Propositions 1 and 2. To prove the sufficiency, we observe that if $\lambda^* = \phi$, then $\lambda^{**} = \omega$. Hence, since $\sigma(\omega, \phi)$ is metrizable and $\sigma(\omega, \phi)$ induces $\sigma(\lambda, \phi)$ on λ, it follows that $\sigma(\lambda, \phi)$ is metrizable.

Remark. If $\langle \Lambda, \Lambda^* \rangle$ is a dual system of Köthe function spaces, the topology $o(\Lambda, \Lambda^*)$ is generated by the family of semi-norms

$$P_g(f) = \int_E |fg|d\mu \qquad (g \in \Lambda^*).$$

We refer the reader to [2] for all definitions and notation regarding Köthe function spaces.

If the lattice operations in Λ are $\sigma(\Lambda, \Lambda^*)$-continuous, then one can show that each $g \in \Lambda^*$ has compact support by methods analogous to those used in Proposition 1. However, if $\Lambda^* = \Phi(E, \mu)$, it does not follow that the lattice operations in Λ are $\sigma(\Lambda, \Lambda^*)$-continuous; in fact, these operations may not even be $\sigma(\Lambda, \Lambda^*)$-sequentially continuous. For example, if E is the unit interval and μ is the Lebesgue measure on E, then $\Phi(E, \mu) = L^\infty[0, 1]$ and $\Omega(E, \mu) = L^1[0, 1]$, and it is easy to show by example that the lattice operations in $L^1[0, 1]$ are not $\sigma(L^1, L^\infty)$-sequentially continuous.

It is a consequence of the general theory of ordered locally convex spaces that the finest compatible topology on λ is the so-called order topology $\mathfrak{T}_o$ (see [9; (4.a), p. 140]). A neighborhood basis of 0 for $\mathfrak{T}_o$ consists of the class of all convex circled subsets of λ that absorb each order interval in λ. This topology is not in general consistent with the dual system $\langle\lambda, \lambda^*\rangle$; that is, the topological dual of $\lambda(\mathfrak{T}_o)$ may not coincide with λ^*. For example, if λ is the space (m) of bounded real sequences, then $\mathfrak{T}_o$ coincides with the usual norm topology on (m) which is strictly finer than the Mackey topology $\tau(\lambda, \lambda^*)$.

An inductive limit characterization of $\mathfrak{T}_o$ obtained by H. Schaefer (see [8; (4.4), p. 134]) has proved to be quite useful in the study of this topology. The following results provide a different, more concrete, inductive limit representation for the order topology on λ if λ is a solid sequence space.

Suppose that λ is a sequence space and that y is a given element of the positive cone K in λ. Then, as we have noted in Section 2, λ_y^* is order isomorphic either to (m), or ϕ, or the product space $(m) \times \phi$. The order topology on (m) is generated by the standard supremum norm on that space, while the order topology on ϕ coincides with the locally convex sum topology if ϕ is regarded as the direct sum of countably many copies of the real line. Since the order topology on $m \times \phi$ is the product of the order topologies on (m) and ϕ, we obtain the following result:

PROPOSITION 3. *If* $y = (y_i)$ *is an element of the positive cone* K *in a sequence space* λ, *then* λ_y^*, *equipped with its order topology* $\mathfrak{T}_o^{(y)}$, *is topologically isomorphic to*

(a) (m) *equipped with the supremum norm if all but a finite number of components of* y *are non-zero,*

(b) ϕ *equipped with its locally convex sum topology if all but a finite number of the components of* y *are zero,*

(c) *the product of* (m) *equipped with the supremum norm and* ϕ, *equipped with the locally convex sum topology, if* y *has an infinite number of zero as well as non-zero components.*

Now suppose that λ is a solid sequence space and that H is an exhausting subset of the positive cone K in λ; that is, for each $x \in K$, there exist $y \in H$ and $\alpha > 0$ such that $x \leq \alpha y$. The family $\{\lambda_y^*: y \in H\}$ is directed since

$$y \leq z \text{ implies } \lambda_y^* \subset \lambda_z^* .$$

Moreover, if $y, z \in H$ and $y \leq z$, the embedding map of λ_y^* into λ_z^* is continuous for the corresponding order topologies $\mathfrak{T}_o^{(y)}$ and $\mathfrak{T}_o^{(z)}$ since it is a positive linear mapping (see [5; (5.2)]). Since λ is solid,

$$\lambda = \bigcup_{y \in H} \lambda_y^*;$$

hence we can define the inductive limit topology $\mathfrak{T}$ on λ with respect to the family $\{\lambda_y^*(\mathfrak{T}_o^{(y)}): y \in K\}$ and the corresponding embedding maps I_y of λ_y^* into λ. Since K is $\mathfrak{T}$-normal (see, for example, [5; p. 42, Remark]), we know that $\mathfrak{T}$ is coarser than the order topology $\mathfrak{T}_o$ on λ. On the other hand, $\mathfrak{T}_o$ is coarser than $\mathfrak{T}$ since each embedding map I_y is positive and therefore continuous for $\mathfrak{T}_o^{(y)}$ and $\mathfrak{T}_o$. We have now proved the following result:

PROPOSITION 4. *If λ is a solid sequence space and* H *is an exhausting subset of the cone* K *in λ, then λ, equipped with its order topology, is the inductive limit of the family $\{\lambda_y^*(\mathfrak{T}_o^{(y)}): y \in H\}$ of subspaces of λ, equipped with their respective order topologies.*

COROLLARY. *If λ is a solid sequence space with an order unit* e, *then $\lambda(\mathfrak{T}_o)$ is topologically isomorphic to* (m) *equipped with the supremum norm.*

Proof. This is an immediate consequence of Propositions 3, 4 and the fact that $e_i > 0$ for all i if $e = (e_i)$ is an order unit.

Remark. Proposition 4 carries over immediately to Köthe function spaces since its proof depends only on the properties of the order topology. The result is that $\Lambda(T_o)$ is the inductive limit of the spaces $\{L_g^\infty(\mathfrak{T}_o^{(g)}): g \in \Lambda,\ g \geq 0\}$ (see [2; p. 100]).

PROPOSITION 5. *If λ is a sequence space and λ^* contains an order unit* e, *then λ is order isomorphic to a $\sigma(\ell^1, (m))$-dense linear subspace of ℓ^1.*

Proof. Since e is an order unit, each component e_i of e is positive. For each $x \in \lambda^{**}$ and each $v \in \lambda^*$, define $\sigma(x) = (e_i x_i)$ and $\rho(v) = (v_i/e_i)$; then $\sigma(x) \in \ell^1$ and $\rho(v) \in (m)$ since e is an order unit in λ^*. It is easy to verify that σ and ρ are positive, linear, and one-to-one. If $y \in \ell^1$ and $v \in \lambda^*$, define $x = (x_i)$ by $x_i = y_i/e_i$ $(i = 1, 2, \cdots)$. Then

$$\sum |x_i v_i| = \sum |y_i(v_i/e_i)| < +\infty$$

since $(v_i/e_i) = \rho(v) \in (m)$; hence σ maps λ^{**} onto ℓ^1.

By making use of a similar argument, we can show that ρ maps λ^* onto (m). It is clear that $\langle x, v\rangle = \langle\sigma(x), \rho(v)\rangle$ for each $x \in \lambda^{**}$ and each $v \in \lambda^*$; hence λ^{**} is order isomorphic to ℓ^1. Since λ is $\sigma(\lambda^{**}, \lambda^*)$-dense in λ^{**}, λ is order isomorphic to the $\sigma(\ell^1, (m))$-dense subspace $\sigma(\lambda)$ of ℓ^1. This completes the proof.

The preceding result will be helpful in drawing certain conclusions concerning bounded linear operators on sequence spaces. Recall that a linear operator T on a locally convex space $E(\mathfrak{T})$ is *bounded* if it maps some $\mathfrak{T}$-neighborhood of 0 into a bounded set. Clearly every bounded linear operator is continuous.

PROPOSITION 6. *If* T *is a positive linear operator on λ and λ is equipped with* $o(\lambda, \lambda^*)$, *then* T *is bounded if and only if* T *is $\sigma(\lambda, \lambda^*)$-continuous and λ^* contains an element which is an order unit for the range of the adjoint operator* T'.

Proof. If T is bounded, then T is $o(\lambda, \lambda^*)$-continuous; hence it is also $\sigma(\lambda, \lambda^*)$-continuous since $o(\lambda, \lambda^*)$ is consistent with the dual system $\langle \lambda, \lambda^* \rangle$ (see [1; Chapter IV, Section 4, Prop. 6, p. 103]). Moreover, there is an element u_o in the cone K' of λ^* such that for each $u \in K'$, one can find a constant $C_u > 0$ for which

$$T([-u_o, u_o]^o) \subset C_u[-u, u]^o.$$

But then if $v \in \lambda^*$, we obtain the result

$$T'v \in T'([-|v|, |v|]) \subset C_{|v|}[-u_o, u_o],$$

that is, $T'v \leq C_{|v|}u_o$. Thus u_o is an order unit for the range of T'.

Conversely, suppose that T is $\sigma(\lambda, \lambda^*)$-continuous and that $u_o \in \lambda^*$ is an order unit for the range of T'. Then for each $u \in \lambda^*$, there exists a positive constant C_u such that $T'u \leq C_u u_o$. Given $u \in K'$, define $C'_u = \max(C_u, C_{-u})$; then

$$T'[-u, u] \subset C'_u[-u_o, u_o]$$

since T' is a positive linear mapping. It follows that

$$T([-u_o, u_o]^o) \subset C'_u[-u, u]^o;$$

hence T is bounded for $o(\lambda, \lambda^*)$.

COROLLARY. *If* λ^* *contains an order unit, then every positive linear operator* T *on* λ *is bounded for* $o(\lambda, \lambda^*)$ *provided that one of the following conditions is satisfied:*

(1) λ *is perfect,*

(2) T *is* $\sigma(\lambda, \lambda^*)$*-continuous.*

Proof. If condition (2) is satisfied, the assertion is an immediate consequence of Proposition 6. If (1) holds, then with respect to the topology $o(\lambda, \lambda^*)$, λ is complete (see [4; Section 30,5(7), p. 416]), the lattice operations in λ are continuous, and the cone K in λ is normal. Moreover, since λ^* contains an order unit, $o(\lambda, \lambda^*)$ is normable, (see [6; Prop. 1.3, p. 204]). The following result is due to H. Schaefer but does not appear explicitly in his published works: If $E(\mathfrak{T})$ is a complete metrizable locally convex space which is a vector lattice with a normal cone and $\mathfrak{T}$-continuous lattice operations, then $\mathfrak{T}$ coincides with the order topology $\mathfrak{T}_o$. A proof of this fac can be found in [7].

It follows from this result that $o(\lambda, \lambda^*)$ coincides with the order topology $\mathfrak{T}_o$ on λ. The fact that T is positive then implies that T is $o(\lambda, \lambda^*)$-continuous; hence condition (2) is satisfied.

Remark. It is clear from the proof of Proposition 6 that this result holds for arbitrary dual systems of ordered vector spaces and, in particular, for dual systems $\langle \Lambda, \Lambda^* \rangle$ of Köthe function spaces. The corollary, as well as its proof, also carries over immediately to Köthe function spaces.

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