THE LATTICE-ORDERED GROUP OF AUTOMORPHISMS
OF AN ORDERED SET

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1. INTRODUCTION

Let S be a totally ordered set, and let G be the group of all functions $f: S \to S$ such that f is one-to-one, onto, and the inequality $x < y$ $(x, y \in S)$ implies that $xf < yf$. We call such a function an *automorphism* of S. If f and g are automorphisms of S, then define $f \leq g$ if $xf \leq xg$ for all $x \in S$. It is well known and easily proved that this defines a partial order on G under which G is a lattice-ordered group (ℓ-group). For example, Problem 95 in [2] asks what ℓ-groups can be constructed in this way. In Section 2 we give a partial answer to this question by showing (Theorem 2) that any ℓ-group can be embedded in the ℓ-group of automorphisms of an appropriate ordered set. (*Ordered* means here, and throughout the paper, *totally ordered*.) The main embedding theorem (Theorem 1) gives more precise information on the embedding and suggests a more concrete formulation of Birkhoff's problem as follows: What ℓ-groups are *transitive* groups of automorphisms of ordered sets? The answer to this question is given by Theorem 3. Section 3 contains an application of the main embedding theorem. We prove that every ℓ-group can be embedded in a divisible ℓ-group. In Section 4, as an illustration of the techniques involved, we investigate the structure of the ℓ-group of permutations of the real line.

Notation. All groups will be written multiplicatively, and (most) functions will be written on the right. Thus if G is a group of permutations of a set S, if $f, g \in G$, and if $x \in S$, then fg is the function whose value at x is sometimes denoted by g(f(x)).

2. THE EMBEDDING THEOREM

Lemmas 1 through 4 are generalizations of lemmas that are well known if the subgroups under consideration are ℓ-ideals. In particular, the proofs of Lemmas 1 and 2 are sufficiently similar to the standard proofs (Birkhoff [2]) that they are omitted here.

Throughout the paper, G denotes an ℓ-group. A subgroup of G, which is also a sublattice, is an ℓ-*subgroup*. A subgroup C of G is *convex*, provided C contains along with any $x \geq 1$ also all y such that $x \geq y \geq 1$. For $x \in G$, $|x| = x \vee x^{-1}$. (The symbols $\vee$ and $\wedge$ denote the lattice operations.)

LEMMA 1. *Let* C *be a convex* ℓ-*subgroup of* G, *and let* $1 \leq a \in G$. *Define* $C^*(a) = \{x \in G \mid a \wedge |x| \in C\}$. *Then* $C^*(a)$ *is a convex* ℓ-*subgroup of* G *and* $C \subseteq C^*(a)$.

LEMMA 2. *Let* C *be a convex subgroup of* G. *Let* $R(C) = \{Cg \mid g \in G\}$ *be the set of all right cosets of* C *in* G. *If we define* $Cg \leq Ch$ *to mean there exists* $c \in C$

with cg $\leq$ h, *then this defines a partial order on the set* R(C). *If* C *is a sublattice,* R(C) *is a lattice with* Cx $\vee$ Cy = C(x $\vee$ y).

In what follows, the only order put on R(C) is that described in Lemma 2.

LEMMA 3. *Let* C *be a convex* ℓ-*subgroup of* G. *The set of convex* ℓ-*subgroups of* G *which contain* C *form a tower (under inclusion) if and only if* a $\wedge$ b = 1 *implies* a $\in$ C *or* b $\in$ C.

Proof. Suppose a, b $\notin$ C and a $\wedge$ b = 1. Then b $\in$ C*(a) $\diagdown$ C*(b) and a $\in$ C*(b) $\diagdown$ C*(a). Moreover, C*(a) $\supset$ C $\subset$ C*(b).

Conversely, if A $\supset$ C $\subset$ B with A and B convex ℓ-subgroups of G and if a' $\in$ A $\diagdown$ B and b' $\in$ B $\diagdown$ A, where a', b' $\geq$ 1, then 1 $\leq$ a' $\wedge$ b' $\leq$ a', b'. Denote set-theoretic union and intersection by $\cup$ and $\cap$ respectively. Then since A and B are convex, a' $\wedge$ b' $\in$ A $\cap$ B. Let a = a'(a' $\wedge$ b')$^{-1}$ and b = b'(a' $\wedge$ b')$^{-1}$. Then a $\wedge$ b = 1, but a, b $\notin$ A $\cap$ B $\supseteq$ C.

LEMMA 4. *If* C *is a convex* ℓ-*subgroup of* G *and if the set of convex* ℓ-*subgroups of* G *which contain* C *is a tower, then* R(C) *is totally ordered.*

Proof. By way of contradiction, if neither Cg $\leq$ Cf nor Cf $\leq$ Cg, then g' = g(g $\wedge$ f)$^{-1}$ $\notin$ C since otherwise Cg = Cg $\wedge$ f $\leq$ Cf. Likewise, f' = f(g $\wedge$ f)$^{-1}$ $\notin$ C. But g' $\wedge$ f' = 1. Hence by Lemma 3, the convex ℓ-subgroups of G which contain C do not form a tower.

LEMMA 5. *Let* C *be a convex* ℓ-*subgroup of* G, *and let* C' *be a convex* ℓ-*subgroup of* G *which properly contains* C *and such that every convex* ℓ-*subgroup of* G *which properly contains* C *also contains* C'. *Then the set of convex* ℓ-*subgroups of* G *which contain* C *form a tower.*

Proof. If 1 $\leq$ k $\in$ C' $\diagdown$ C, then since k $\notin$ C*(k), it follows that C*(k) = C. Now let a $\wedge$ b = 1. Suppose a $\notin$ C. Then since a $\in$ C*(b), C*(b) properly contains C, and therefore C*(b) $\supseteq$ C'. Let 1 < k $\in$ C' $\diagdown$ C. Then b $\in$ C*(k). Hence b $\in$ C. Thus by Lemma 3, the set of convex ℓ-subgroups of G which contain C form a tower.

The proof of the following lemma is straight-forward.

LEMMA 6. *Let* C *be a convex subgroup of* G, *and suppose* R(C) *is totally ordered. Then each* g $\in$ G *induces an automorphism* β(g, C) *of* R(C) *defined by*

$$(Cx)\beta(g, C) = Cxg.$$

If C is a convex subgroup of G and if R(C) is totally ordered, we let A(C) denote the ℓ-group of automorphisms of R(C).

LEMMA 7. *If* C *is a convex* ℓ-*subgroup of* G *and if* R(C) *is totally ordered, then the mapping* α(C): G $\rightarrow$ A(C) *defined by* gα(C) = β(g, C) *is an* ℓ-*group homomorphism of* G *onto a transitive* ℓ-*subgroup of* A(C).

Proof. The only non-trivial part of the proof is to show that the lattice operations are preserved. We must show that

$$(g \vee 1)\alpha(C) = \beta(g, C) \vee i$$

where i denotes the identity function in A(C).

In other words, we must show that for any right coset Cx,

$$Cx(g \vee 1) = (Cxg) \vee (Cx).$$

But this follows immediately from Lemma 2.

LEMMA 8. *If* C[g] *is a maximal element of the set of convex ℓ-subgroups of* G *which do not contain* 1 ≠ g ∈ G, *then the set of convex ℓ-subgroups of* G *which contain* C[g] *forms a tower.*

Proof. Let K be the intersection of all convex ℓ-subgroups of G which contain C[g] and g. Then every convex ℓ-subgroup of G which properly contains C[g] also contains K. The result now follows from Lemma 5.

THEOREM 1. *(Main Embedding Theorem). If* G *is an ℓ-group, then* G *is ℓ-isomorphic to a subdirect sum of ℓ-groups* { B[g] | 1 ≠ g ∈ G } *such that each* B[g] *is a transitive ℓ-subgroup of the ℓ-group of automorphisms of a totally ordered set* S[g].

Proof. For each 1 ≠ g ∈ G there exists, by Zorn's lemma, a convex ℓ-subgroup C[g] of G which is maximal without g. By Lemma 8, the set of convex ℓ-subgroups of G which contain C[g] forms a tower. By Lemma 4, S[g] = R(C[g]) is totally ordered. By Lemma 7, the mapping α(C[g]): G → A(C[g]) is an ℓ-group homomorphism of G onto a transitive ℓ-subgroup B[g] of A(C[g]). Moreover, g is not in the kernel of α(C[g]) since C[g]g ≠ C[g]. The theorem now follows from the standard results (see Birkhoff [2]).

If H is the direct sum of ℓ-groups B_α and if each B_α is the ℓ-group of automorphisms of an ordered set S_α where $S_\alpha \cap S_\beta = \emptyset$ for $\beta \neq \alpha$, then we may totally order the set $\bigcup S_\alpha$ as follows: first order the collection of sets S_α in any way; for example, it may be well-ordered. Then for x, y ∈ $\bigcup S_\alpha$, call x < y if x, y ∈ S_α and x < y as elements of S_α, or if x ∈ S_α and y ∈ S_β where $S_\alpha < S_\beta$. If ϕ ∈ H then ϕ induces an automorphism of the set $\bigcup S_\alpha$ ordered in this way, as follows: xϕ' = xϕ_α, where x ∈ S_α and ϕ_α is the αth component of ϕ. From this and Theorem 1, we have the following theorem.

THEOREM 2. *If* G *is an ℓ-group,* G *is ℓ-isomorphic to an ℓ-subgroup of the ℓ-group of automorphisms of an ordered set.*

The next theorem describes those ℓ-groups for which the embedding of Theorem 1 can be chosen so that there is only one summand.

THEOREM 3. *An ℓ-group* G *is ℓ-isomorphic to a transitive ℓ-subgroup of the ℓ-group of automorphisms of an ordered set if and only if there exists a convex ℓ-subgroup* C *of* G *such that both*

(1) *the set of convex ℓ-subgroups of* G *containing* C *is totally ordered under inclusion, and*

(2) *the only ℓ-ideal of* G *contained in* C *is* {1}.

Proof. If G is a transitive ℓ-subgroup of the ℓ-group of automorphisms of an ordered set L, and if x ∈ L, then C = {g ∈ G | xg = x} is clearly a convex ℓ-subgroup of G. C contains no ℓ-ideals of G. For if 1 ≠ g ∈ C, say yg ≠ y, then, as G is transitive, there exists f ∈ G such that xf = y. Therefore

$$xfgf^{-1} = ygf^{-1} \neq yf^{-1} = x,$$

so $fgf^{-1} \notin$ C. The ℓ-ideals of G containing C form a tower, for otherwise, by Lemma 3, there exist a, b $\notin$ C such that a ∧ b = 1. That is, xa ≠ x ≠ xb, and yet

$$x = x1 = x(a \wedge b) = (xa) \wedge (xb),$$

which is impossible since L is totally ordered.

Conversely, if C is such a subgroup of G, then by Lemma 4, R(C) is totally ordered, and by Lemma 7, the mapping α(C) is a homomorphism of G onto a transitive subgroup of A(C). If g is in the kernel of α(C), then Cg = C; thus the kernel is contained in C. As the kernel is an ℓ-ideal of G, the kernel is $\{1\}$ and α(C) is one-to-one.

COROLLARY 1. *If there exists an ℓ-ideal* K $\neq$ $\{1\}$ *of* G *such that every non-trivial ℓ-ideal of* G *contains* K, *then* G *is a transitive ℓ-group of automorphisms of an ordered set.*

Proof. Let $1 \neq g \in K$, and let C[g] be a convex ℓ-subgroup of G maximal without g. Then C[g] satisfies conclusions (1) and (2) of the theorem.

COROLLARY 2. *A simple ℓ-group (without non-trivial ℓ-ideals) is a transitive ℓ-group of automorphisms of an ordered set.*

COROLLARY 3. *If* G *is abelian and is a transitive ℓ-group of automorphisms of an ordered set, then* G *is totally ordered.*

Proof. Any such C is an ℓ-ideal. Hence C = $\{1\}$, and G is isomorphic as an ordered set to R(C), which is totally ordered.

In this connection, Cohn [3] proves that a group satisfying certain completeness conditions, which is a group of automorphisms of an ordered set, admits a total order if and only if it is abelian.

3. REMARKS ON GROUPS OF AUTOMORPHISMS

Most of the elementary properties of ℓ-groups are almost self-evident for ℓ-groups of automorphisms of ordered sets. To show this, as well as to establish some notation for the next section, we consider an ℓ-group G of automorphisms of an ordered set L. Let $g \in G$ be given. For any $x \in L$ let

$$I(x) = \{y \in L \mid \text{there exist integers } m, n \text{ such that } xg^n \leq y \leq xg^m\} .$$

Then I(x) is convex in L, (I(x))g = I(x), and if $y \in I(x)$, then I(y) = I(x). If I(x) contains more than one point then I(x) is a *supporting interval of* g. The union of the supporting intervals of g is the *support* of g. In any case, we call I(x) an *interval of* g. The collection of intervals of g determine an equivalence relation on L (x ~ y if and only if I(x) = I(y)). For a given interval I(x) of g and for all $y \in I(x)$, either (1) yg = y, in which case I(x) = $\{x\}$, and we say g is *zero on* I(x), or (2) yg < y, and we say g is *negative on* I(x), or (3) yg > y, and we say g is *positive on* I(x).

It is clear that if $g^+ = g \vee 1$, then for all $x \in L$,

$$xg^+ = \begin{cases} xg & \text{if g is positive on I(x)} \\ x & \text{otherwise.} \end{cases}$$

Likewise, $g^- = g \wedge 1$ is such that

$$xg^- = \begin{cases} xg & \text{if g is negative on I(x)} \\ x & \text{otherwise.} \end{cases}$$

Similarly, $|g| = g \vee g^{-1}$ is such that

$$x|g| = \begin{cases} xg & \text{if } g \text{ is positive on } I(x) \\ xg^{-1} & \text{otherwise.} \end{cases}$$

Moreover, for $1 \leq g$, $h \in G$, $g \wedge h = 1$ if and only if the support of g does not meet the support of h.

Thus the following elementary properties of ℓ-groups are obvious for these groups of automorphisms:

(1) $|g| > 1$ for all $g \neq 1$;

(2) If $g \wedge h = 1$, then $g \vee h = gh = hg$;

(3) If $g^n \geq 1$, then $g \geq 1$.

We note also that if G is the ℓ-group of *all* automorphisms of L, then G is "laterally complete" in the sense that any collection of pairwise disjoint positive elements of G has a least upper bound. It follows from Theorem 2 that any ℓ-group can be embedded in a laterally complete ℓ-group. This generalizes an example of Birkhoff [2; p. 242, Example 2(b)].

4. AN APPLICATION

An ℓ-group G is *divisible* provided that for all $g \in G$ and for all integers $n > 0$, there exists an $h \in G$ such that $h^n = g$. We use the theory of Section 2 to show that every ℓ-group can be embedded in a divisible ℓ-group.

A totally ordered set N is called an η_α *set* (Hausdorff [6]) if for every pair of subsets A and B of N such that $|A|$, $|B| < \aleph_\alpha$ and such that every element of A is less than every element of B $(A < B)$, there exists an $x \in N$ such that $A < x < B$. An η_α set of cardinality $\aleph_\alpha$ will be called an α-*set*. It is known that any two α-sets are isomorphic, that any α-set contains every ordered set of smaller cardinality, and that α-sets of arbitrarily large cardinality exist (assuming the generalized continuum hypothesis). These α-sets play a role in "universal embedding theorems" for totally ordered abelian groups [1] and totally ordered fields [5].

THEOREM 4. *Let* N *be an* α-*set and let* G *be the* ℓ-*group of all automorphisms of* N. *Then* G *contains as an* ℓ-*subgroup every* ℓ-*group of cardinality less than* $\aleph_\alpha$.

Proof. It is clear from the proofs of Theorems 2 and 3 that any ℓ-group of cardinality less than $\aleph_\alpha$ can be embedded in an ℓ-group K such that K is the ℓ-group of automorphisms of an ordered set L and L has cardinality less than $\aleph_\alpha$. We now show that any such K can be embedded in G.

As L can be embedded in N, we consider $L \subset N$. Now define an equivalence relation on N as follows: for x, y $\in$ N, let xEy if the intervals [x, y) and [y, x) contain no element of L. Then the equivalence classes are convex, and thus N/E has a natural total order. Also, each equivalence class contains at most one element of L; and if an equivalence class contains an element of L, then that element is the upper end point of the equivalence class. An equivalence class is said to be of type 1 if it contains an element of L, otherwise, it is of type 2.

It is not hard to verify that any class of type 2 is an α-set, and that any class of type 1 minus its end point is also an α-set. Hence any two classes of type 1 are isomorphic and any two classes of type 2 are isomorphic. Now for each pair of E-classes A and B of the same type, choose an isomorphism γ(A, B) of A onto B such that if A, B, and C are all of the same type then γ(A, B)γ(B, C) = γ(A, C). This is clearly always possible; for example, let C be a fixed E-class of type 1 and choose an arbitrary collection of isomorphisms γ(A, C) for all A of type 1, A $\neq$ C. Then let γ(C, C) be the identity, and for all A and B, let

$$\gamma(A,\ B) = \gamma(A,\ C)(\gamma(B,\ C))^{-1}\ .$$

Do a similar thing for the E-classes of type 2.

Let ϕ be an automorphism of L. Since each E-class contains at most one element of L, L is isomorphic to the subset L' = LE of N/E. Thus ϕ induces a natural isomorphism ϕ' of L'. It is easy to verify that every cut of L' determines exactly one element of (N/E); that is, if A $\cup$ B = L' and A $<$ B then there exists exactly one element n of N/E such that A $<$ n $\leq$ B. We may now extend ϕ' to an automorphism ϕ'' of N/E as follows: if x $\in$ L' then xϕ'' = xϕ', and if x $\in$ (N/E)$\diagdown$L' then x determines a cut A $<$ x $<$ B of L'; the cut Aϕ' $<$ Bϕ' of L' determines a unique y $\in$ (N/E)$\diagdown$L' such that Aϕ' $<$ y $<$ Bϕ'; let xϕ'' = y, and it follows that ϕ'' is an isomorphism of N/E.

We now extend ϕ to an automorphism ϕ^* of N as follows: for all x $\in$ N,

$$x\phi^* = x\gamma(xE,\ (xE)\phi'')\ .$$

It is a straight-forward matter to verify that the mapping $\phi \rightarrow \phi^*$ is an ℓ-isomorphism of K into G.

LEMMA 9. *Let* S *be an ordered set in which any two non-trivial closed intervals are isomorphic. Then the ℓ-group of all automorphisms of* S *is divisible.*

Proof. Let g $\in$ G and a positive integer n be given, where G is the ℓ-group of all automorphisms of S. We wish to find an f $\in$ G such that f^n = g. There is no loss of generality in assuming that g is positive and has only one supporting interval (see Section 3). It follows from the hypothesis that S is dense in itself; that is, if x, y $\in$ S and x $<$ y, then there exists a z $\in$ S such that x $<$ z $<$ y.

Let $a_0 < a_0 g$ for some $a_0 \in$ S. Choose

$$a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = a_0 g\ .$$

Since any two non-trivial closed intervals in S are isomorphic, it follows that there exist isomorphisms

$$\phi_i:\ (a_{i-1},\ a_i] \rightarrow (a_i,\ a_{i+1}]\qquad (i = 1,\ 2,\ \cdots,\ n - 1)\ .$$

Define $\phi_n:\ (a_{n-1},\ a_n] \rightarrow (a_n,\ a_1 g]$ by

$$x\phi_n = x\phi_{n-1}^{-1}\phi_{n-2}^{-1}\cdots\phi_1^{-1}g\ .$$

Now let $\phi^*:\ (a_0,\ a_n] \rightarrow (a_1,\ a_1 g]$ be the extension of all the ϕ_i, for i = 1, 2, $\cdots$, n. Then ϕ^* is an isomorphism.

Define $f \in G$ as follows: if x is not in the support of g, then $xf = x$; if x is in the support of g, then there is a unique integer $m(x)$ (not necessarily positive) such that

$$x \in (a_0 \, g^{m(x)}, \, a_n \, g^{m(x)} = a_0 \, g^{m(x)+1}];$$

in this case, let $xf = xg^{-m(x)} \phi^* g^{m(x)}$. Then f is an automorphism of S and $f^n = g$.

THEOREM 5. *Every ℓ-group can be embedded in a divisible ℓ-group.*

Proof. Since there exist α-sets of arbitrarily large cardinality, it follows from Theorem 4 that every ℓ-group can be embedded in the ℓ-group of automorphisms of some α-set. If $[a, b]$ is a closed interval in an α-set, then it is easily seen that (a, b) is an α-set, unless $a \geq b$. Hence any two non-trivial closed intervals in an α-set are isomorphic. Thus by Lemma 9, the ℓ-group of automorphisms of an α-set is divisible.

5. AUTOMORPHISMS OF THE REAL LINE

In order that the embedding theorem (Theorem 1) shed much light on the structure of ℓ-groups, it would be necessary to know how the structure of an ordered set affects the structure of any transitive ℓ-group of automorphisms of the set, or conversely, to what extent a given ℓ-group determines the sets over which it is a transitive ℓ-group of automorphisms. These questions are not answered in this paper; however, the ℓ-group of automorphisms of the real line seems to illustrate many of the possibilities, and in this section we investigate this ℓ-group in some detail. Most of the results in this section can be generalized. For example, Lemma 10 remains true under the hypothesis of Lemma 9. For other results on this group, see Everett and Ulam [4].

Throughout this section, G is the ℓ-group of all automorphisms of the real line.

LEMMA 10. *Two elements g, f $\in$ G are conjugates if and only if there exists a one-to-one function ϕ from the set of intervals of g onto the set of intervals of f such that for all intervals I and J of g,*

(1) *if I < J (i $\in$ I, j $\in$ J implies i < j), then Iϕ < Jϕ, and*

(2) *if g is positive (negative, zero) on I, then f is positive (negative, zero) on Iϕ.*

Proof. Suppose $f = h^{-1}gh$. Then let $I\phi = Ih$ for each interval I of g.

Conversely, let such a ϕ exist. Define $h \in G$ as follows: If $x \in R$, the reals, then x belongs to exactly one of the intervals of g, say $x \in I$. If g is zero on I, let xh be the only element of Iϕ (by assumption, f is zero on Iϕ). If g is positive on I (and similarly if g is negative on I), choose $x_0 \in I$ and $y_0 \in I\phi$. Then there exists $k \in G$ such that $x_0 k = y_0$ *and* $x_0 gk = y_0 f$. Also there exists a unique integer $n(x)$ such that $x \in (x_0 \, g^{n(x)}, \, x_0 \, g^{n(x)+1}]$. Let $xh = xg^{-n(x)}kf^{n(x)}$. Then it is easy to verify that h is an automorphism of R and that $f = h^{-1}gh$.

It is a corollary (due to T. Lloyd) that, for every $g \in G$, g and g^2 are conjugates; and so g is a commutator.

THEOREM 6. *Let*

$A = \{g \in G \mid \text{ there exists an } x_g \in R \text{ such that } y \geq x_g \text{ implies } yg = y\}$;

$B = \{g \in G \mid \text{ there exists a } y_g \in R \text{ such that } y \leq y_g \text{ implies } yg = y\}$;

$C = A \cap B.$

Then A, B, *and* C *are the only proper ℓ-ideals of* G. *Moreover,* C *is the only proper ℓ-ideal of* A *or* B, *and* C *has no proper ℓ-ideals.*

Proof. Let $1 < g \notin A$, and let $1 < b \in B$, say $yb = y$ for all $y \leq y_0$. We now show that any ℓ-ideal of G or of B which contains g also contains b.

First suppose g has no fixed point above some $x_0 \in R$. There exists $p \in B$ such that $x_0 p = y_0$. Then $p^{-1}gp$ has no fixed points above y_0. Hence the set of intervals of $p^{-1}gp$ is the same as the set of intervals of $g' = (p^{-1}gp) \vee b$. Thus from Lemma 10, $p^{-1}gp$ and g' are conjugate; in fact, since $p^{-1}gp$ and g' agree on all $x \leq y_0$, a glance at the proof of Lemma 10 shows that $p^{-1}gp$ and g' are conjugate by an element of B. Hence g and g' are conjugate by an element of B. Moreover, $g' \geq b \geq 1$. Therefore, any ℓ-ideal of G which contains g also contains b; and any ℓ-ideal of B which contains g also contains b.

The other case is if g has arbitrarily large fixed points. Then there is a sequence of supporting intervals $I_i = (a_i, b_i)$ of g such that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_i < b_i < \cdots$$

and $\{a_i\}$ is cofinal in R. There exists a $q \in B$ such that

$$a_i q = 4(i - 1) \text{ and } b_i q = 4i - 1.$$

Each of the intervals $(4(i - 1), 4i - 1)$ $(i = 1, 2, \cdots)$ is a supporting interval of the function $q^{-1}gq$. Likewise, there exists an $r \in B$ such that

$$a_i r = 4i - 2 \text{ and } b_i r = 4i + 1.$$

Each of the intervals $(4i - 2, 4i + 1)$ $(i = 1, 2, \cdots)$ is a supporting interval of the function $r^{-1}gr$. Hence

$$g'' = (q^{-1}gq) \vee (r^{-1}gr)$$

has no fixed points above 0. Now returning to the first case with g'' in place of g, we again obtain the conclusion that any ℓ-ideal of G or of B which contains g must also contain b. Thus any ℓ-ideal of G which is not contained in A must contain B. By symmetry we may interchange A and B in the previous sentence. But it is easily seen that A and B together generate G. Thus every proper ℓ-ideal of G is contained in A or in B. But we have also shown above that every proper ℓ-ideal of B (or A, by symmetry) is contained in C.

Finally, we show that C contains no proper ℓ-ideals. Let g and f be any positive elements of C; say, $yf = y$ for all $y \geq y_0$ and for all $y \leq x_0$. Let (a, b) be a supporting interval of g. Then there exists an $s \in C$ such that $as = x_0 - 1$ and $bs = y_0 + 1$. Letting $h = s^{-1}gs$, we see that, for some positive integer n, $y_0 < x_0 h^n$. Hence for all $x \in [x_0, y_0]$, $xf \leq y_0 < x_0 h^n \leq xh^n$. And for all other $x \in R$, $xf = x \leq xh^n$. Hence any ℓ-ideal of C which contains g also contains f. It follows that C has no proper ℓ-ideals. Thus, the theorem is proved.

T. Lloyd has shown that A, B, and C are, in fact, the only *normal* subgroups of G, and C is algebraically simple. (This last result is also contained in [7].)

The ℓ-group C and the four mutually isomorphic ℓ-groups G/A, G/B, A/C, and B/C are examples of simple ℓ-groups which are not totally ordered. This answers in the negative a conjecture of Lorenz [8].

With only minor alterations in the proof, Theorem 6 is also true for the ℓ-groups of automorphisms of the rationals and irrationals, respectively. In fact, these two ℓ-groups are identical, for any automorphism of the rationals can be extended uniquely to an automorphism of the reals, and this cuts down to an automorphism of the irrationals. The correspondence is an isomorphism.

The respective ℓ-groups of automorphisms of the rationals and the reals differ, however, in an important way. A sequence $f_1 < f_2 < \cdots$ of elements of an ℓ-group H is o-*regular* (Everett and Ulam [4], also Birkhoff [2; p. 232]) if there exists a sequence $v_1 \geq v_2 \geq \cdots$ of elements of H such that $\wedge_n v_n = 1$ and for all positive integers n and p, $f_{n+p} f_n^{-1} \leq v_n$ and $f_n^{-1} f_{n+p} \leq v_n$. An o-regular sequence $\{f_i\}$ *converges to* f if $f = \vee_i f_i$. H is o-*complete* if every o-regular sequence in H converges.

THEOREM 7. *Let* H *be the* ℓ-*group of automorphisms of an ordered set* L *such that*

(1) L *is relatively complete, and*

(2) *every interval of* L *which contains more than one point contains the support of some element of* H.

Then H *is* o-*complete*.

Proof. Let $f_1 < f_2 < \cdots$ be an o-regular sequence in H with $\{v_i\}$ as above. Consider the function f: L $\rightarrow$ L defined by

$$xf = \bigvee_i (xf_i).$$

As L is relatively complete and $xf_i \leq xv_1 f_1$, f is well defined. Also, if x, y $\in$ L and $x \leq y$, then $xf \leq yf$. Thus if f is one-to-one and onto, then f $\in$ H and $\{f_i\}$ converges to f.

Suppose, by way of contradiction, that f is not one-to-one; say, xf = yf for some $x < y$. Let $1 < v \in$ H such that the support of v is contained in the interval [x, y]. For any positive integer n,

$$yf_n \leq yf = \bigvee_m (xf_m).$$

It follows that for any $z < yf_n$ there exists a positive integer p such that $z < xf_{n+p}$. Hence

$$zf_n^{-1} < xf_{n+p} f_n^{-1} \leq xv_n.$$

Therefore,

$$xv_n \geq \bigvee_z (zf_n^{-1}) = (\bigvee_z z)f_n^{-1} = yf_n f_n^{-1} = y.$$

It follows that $v_n > v$. Thus $\wedge_n v_n \geq v > 1$, which is a contradiction. Hence f is one-to-one.

In a like manner, it can be shown that the function g: L $\rightarrow$ L defined by $xg = \wedge_i (xf_i^{-1})$ is one-to-one. If S $\subset$ L is the range of g, then g^{-1}: S $\rightarrow$ L is an onto map. But it is easily seen that g^{-1} and f agree on S. Therefore, f maps onto L.

COROLLARY. *The ℓ-group* G *of automorphisms of the real line is* o-*complete.*

The ℓ-group of automorphisms of the rationals is not o-complete. For if $\{a_i\}$ is a sequence of rationals increasing monotonely to $\sqrt{2}$, if $\{b_i\}$ is a sequence of rationals decreasing monotonely to $\sqrt{2}$, and if we define $xf_i = x + a_i$, then

$$xv_n = x + b_n - a_n > x + a_{n+p} - a_n = xf_n^{-1}f_{n+p} = xf_{n+p}f_n^{-1},$$

and $v_1 > v_2 > \cdots$ with $\wedge_n v_n = 1$. Hence $\{f_i\}$ is o-regular. But if $f = \vee_i f_i$, then for all i,

$$x + a_i \le xf \le x + b_i,$$

which is impossible.

We conclude with some remarks on the Main Embedding Theorem. It is clear that any totally ordered group is a transitive ℓ-group of automorphisms of an ordered set (in its right regular representation, for example). Hence any ℓ-group which is a subdirect sum of ordered groups already satisfies the conclusion of Theorem 1 in a nice way. It is well known that an ℓ-group H is a subdirect sum of ordered groups if and only if H is regular, which is to say that no element of H is disjoint from one of its conjugates (Lorenzen [9]). Thus, for a large class of ℓ-groups, including all abelian ℓ-groups, the embedding of Theorem 1 is of no value. However, very little seems to be known about the structure of non-regular ℓ-groups, and the transitive group of all automorphisms of an ordered set is "almost always" non-regular.

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REFERENCES

1. N. L. Alling, *On ordered divisible groups,* Trans. Amer. Math. Soc. 94 (1960), 498-514.

2. G. Birkhoff, *Lattice theory,* rev. ed., Amer. Math. Soc. Colloq. Publns. vol. 25, New York, 1948.

3. P. M. Cohn, *Groups of order automorphisms of ordered sets,* Mathematika 4 (1957), 41-50.

4. C. J. Everett and S. Ulam, *On ordered groups,* Trans. Amer. Math. Soc. 57 (1945), 208-216.

5. L. Gillman and M. Jerison, *Rings of continuous functions,* Van Nostrand, Princeton, 1960.

6. F. Hausdorff, *Grundzüge der Mengenlehre,* Veit and Co., Leipzig, 1914.

7. G. Higman, *On infinite simple groups,* Publ. Math. Debrecen 3 (1954), 221-226.

8. K. Lorenz, *Über Strukturverbände von Verbandsgruppen,* Acta. Math. Acad. Sci. Hungar. 13 (1962), 55-67.

9. P. Lorenzen, *Über halbgeordnete Gruppen,* Math. Z. 52 (1949), 483-526.

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