CONNEXION PRESERVING, CONFORMAL, AND PARALLEL MAPS

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This paper is a small collection of loosely related results in differential geometry. The methods are perhaps more interesting than the results for they illustrate the power and elegance of completely invariant methods in differential geometry.

Preliminaries. Let M and M' be C^∞ Riemannian manifolds (where we denote the metric tensor by $<X, Y>$), and let $f: M \to M'$ be a C^∞ map. If there exists a C^∞ real-valued function F on M such that for any m in M, then

$$<f_* X, f_* Y> = F(m)<X, Y>$$

for all X, Y in M_m; and if $F > 0$ on M, then f is *conformal*. The map f_* is the differential of f, and f_* has no kernel if f is conformal. We call the map F the *scale function;* notice $F \geq 0$. If F is a constant function, then we say f is *homothetic.* If $F = 1$, we say f is an *isometry,* and if f is both an isometry and a diffeomorphism, we say M is *isometric* to M'.

A *connexion* D on M will be a C^∞ covariant differentiation operator assigning to C^∞ fields X and Y (with common domain A) a C^∞ field $D_X Y$ (on A) such that

$$D_{(X+Z)}Y = D_X Y + D_Z Y ,$$

$$D_X(Z + Y) = D_X Z + D_X Y ,$$

$$D_{fX} Y = fD_X Y ,$$

$$D_X fY = (Xf)Y + fD_X Y ,$$

where Z is a C^∞ field on A and f is a C^∞ real valued function on A. The *torsion* tensor $T(X, Y)$ and *curvature* tensor $R(X, Y)$ of a connexion are defined by

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} .$$

We will use the fact that on a Riemannian manifold there exists a unique (Riemannian) connexion D with zero torsion which satisfies the property

$$X< Y, Z> = <D_X Y, Z> + < Y, D_X Z> ,$$

where Y and Z are vector fields in a neighborhood of the base point of the vector X. If M and M' are C^∞ manifolds with connexions D and D', respectively, then a map $f: M \to M'$ is *connexion preserving* if $f_* D_X Y = D'_{f_*X} f_* Y$ for all vectors X and vector fields Y. (Note $D'_{f_*X} f_* Y$ is well defined since $f_* Y$ is a well-defined

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field on some curve with tangent $f_* X$.) If f is a diffeomorphism of M onto M' and D' is a connexion on M', then there is a unique (induced) connexion D on M for which f is connexion preserving; that is, letting

$$D_X Y = f_*^{-1}(D_{f_* X} f_* Y),$$

one trivially checks that this D satisfies the above properties defining a connexion.

THEOREM 1. *Let* M *and* M' *be* n*-dimensional* C^∞ *Riemannian manifolds with* M *connected, and let* f *be a* C^∞ *conformal map of* M *into* M' *with scale function* F. *Then* f *is (Riemannian) connexion preserving if and only if* f *is homothetic.*

Proof. Suppose that f is connexion preserving, and let us denote both connexions by D (dropping the prime from D'). Take any m in M and any X in M_m. Let Y be a unit vector field on a connected neighborhood U of m. Then

$$f_* X\langle f_* Y, f_* Y\rangle = 2\langle D_{f_* X} f_* Y, f_* Y\rangle = 2\langle f_* D_X Y, f_* Y\rangle$$
$$= 2F(m)\langle D_X Y, Y\rangle = F(m)X\langle Y, Y\rangle = 0.$$

Hence $\langle f_* Y, f_* Y\rangle = F\langle Y, Y\rangle = F$ is a constant function on U. Then since F is C^∞ and M is connected, it follows that F is constant on M.

Conversely, let F be constant on M. We know f_* is non-singular; hence, for any m in M, we may choose neighborhoods U of m and V of f(m) such that f is a diffeomorphism from U onto V. We apply the remark preceding the theorem and let $\overline{D}$ be the connexion induced by f on U. Then we show $\overline{D}$ is the Riemannian connexion on U; and consequently, $D = \overline{D}$ on U, and f is connexion preserving.

Let Y and Z be fields on U and take X in M_m. Then

$$X\langle Y, Z\rangle = (1/F) f_* X\langle f_* Y, f_* Z\rangle$$
$$= (1/F)(\langle D_{f_* X} f_* Y, f_* Z\rangle + \langle f_* Y, D_{f_* X} f_* Z\rangle)$$
$$= (1/F)(\langle f_* \overline{D}_X Y, f_* Z\rangle + \langle f_* Y, f_* \overline{D}_X Z\rangle)$$
$$= \langle \overline{D}_X Y, Z\rangle + \langle Y, \overline{D}_X Z\rangle.$$

Also,

$$\overline{D}_Y Z - \overline{D}_Z Y = f_*^{-1}(D_{f_* Y} f_* Z - D_{f_* Z} f_* Y) = f_*^{-1}[f_* Y, f_* Z] = [Y, Z].$$

Thus $\overline{D}$ is metric preserving and has zero torsion, so it is the Riemannian connexion on U. Q. E. D.

In order to extend the above theorem to connexion-preserving immersions, we make a few remarks on submanifolds and induced connexions. Let $k \leq n$. A k-dimensional C^∞ manifold M is a *submanifold* of an n-dimensional C^∞ manifold $\overline{M}$, if for every point p in M there exists a coordinate system $\bar{x}_1, \cdots, \bar{x}_n$ of $\overline{M}$ with domain $\overline{U}$ such that p in $\overline{U}$ and the set

$$U = [m \text{ in } \overline{U}: \bar{x}_{k+1}(m) = \cdots = \bar{x}_n(m) = 0]$$

is a coordinate neighborhood of p in M for the functions $\bar{x}_1, \cdots, \bar{x}_k$ restricted to U. Thus the inclusion map of M into $\overline{M}$ is C^∞. If $\overline{M}$ is Riemannian with Riemannian connexion $\overline{D}$, M inherits the metric tensor and thus is a Riemannian manifold with Riemannian connexion D. These connexions are related by the Gauss equation

$$\overline{D}_X Y = D_X Y + V(X, Y),$$

where X and Y are fields on M and the right side is the unique decomposition of $\overline{D}_X Y$ into a vector $D_X Y$ tangent to M and a vector V(X, Y) normal to M (for details see [2]). We call V(X, Y) the *second fundamental form* tensor. It is a symmetric 2-covariant vector-valued tensor. If $V \equiv 0$ on M, we say M is *flat* in $\overline{M}$. This is a generalization of the fact that, if M is a closed flat (n - 1)-dimensional submanifold of R^n (real Euclidian n-space), then M is a hyperplane.

A C^∞ map $f: M \to \overline{M}$ is called an *immersion* if f_* is non-singular at all points of M. An immersion is called an *imbedding* if f is univalent. It is easy to show that if $f: M \to \overline{M}$ is an imbedding, then the image set $M' = f(M)$ is a submanifold of $\overline{M}$. By restricting an immersion to a local neighborhood it becomes an imbedding of this neighborhood; hence we say an immersion is flat if all these local imbeddings are flat. We may extend the remarks above concerning the Gauss equation and the second fundamental form tensor to an immersed submanifold; that is, to the image set of an immersion. Henceforth all manifolds and mappings mentioned will be C^∞.

THEOREM 1'. *Let* M *and* $\overline{M}$ *be Riemannian manifolds with* M *connected. Let* f *be a conformal map of* M *into* $\overline{M}$ *with scale function* F. *Then* f *is (Riemannian) connexion preserving if and only if* F *is constant and* f(M) *is flat.*

Proof. Since f_* is non-singular, the dimension of M is less than or equal to the dimension of $\overline{M}$, and f is an immersion of M into $\overline{M}$.

Suppose that f is connexion preserving, so $f_* D_X Y = \overline{D}_{f_* X} f_* Y$ is tangent to $M' = f(M)$. Hence $V(f_* X, f_* Y) = 0$, and so $V = 0$ on M', since f_* is an isomorphism onto the tangent space to M'. Hence $\overline{D}_{f_* X} f_* Y = D'_{f_* X} f_* Y$, where D' is the Riemannian connexion on the flat immersed submanifold M'. Thus f is a conformal connexion-preserving map of M into M', and we may apply Theorem 1 to obtain the conclusion that F is constant on M.

Conversely, if F is constant, then Theorem 1 implies that f is connexion preserving from M to M'. Then M' flat implies that V = 0, so f is connexion preserving from M into $\overline{M}$. Q. E. D.

Corollary. Let M *be a complete, connected,* (n - 1)*-dimensional Riemannian manifold. Let* $f: M \to R^n$ *be a conformal map. The map* f *is connexion preserving if and only if the scale function is constant and* f(M) *is a hyperplane of* R^n.

We next consider the notion of "parallel" submanifolds of R^n. An (n - 1)-dimensional submanifold M of R^n is said to be *framed in* R^n if one has chosen a C^∞ unit normal vector field N on M. This can always be done locally but is only possible globally if M is orientable. Let M be framed in R^n, let $e_1, \cdots, e_n$ be the usual global orthonormal parallel base vectors on R^n, and let $N = \Sigma_1^n a_i e_i$ define C^∞ functions a_i on M. For $r \neq 0$, let $M_r = [p + rN: p \text{ in } M]$; that is, if $p = (p_1, \cdots, p_n)$ is in M, then

$$f(p) = p + rN = (p_1 + ra_1(p), \cdots, p_n + ra_n(p)).$$

The map f is called the *natural map* of M into M_r, and if f is univalent, then M_r is an (n - 1)-dimensional submanifold oriented in R^n, which we call a *parallel submanifold of* M.

In what follows we at times equate two vectors in tangent spaces of R^n at different points, and in that case we mean the vectors have equal components with respect to the base $e_1, \cdots, e_n$.

Recall that, if M is an (n - 1)-dimensional submanifold framed in R^n by N, then we can define a linear transformation L on each tangent space of M by $L(X) = \overline{D}_X N$, where $\overline{D}$ is the Riemannian connexion on R^n. In the above notation, $L(X_m) = \Sigma_1^n (Xa_i)e_i$ at m, and the mapping m into L at m is C^∞. The vector LX is tangent to M since

$$2<\overline{D}_X N, N> = X<N, N> = 0$$

for $<N, N> \equiv 1$ on M. The algebraic invariants of L define the curvature (det L), mean curvature (trace L), principal curvatures (eigenvalues of L), and principal directions (eigenvectors of L). All the above exist since L is self-adjoint; that is,

$$<LX, Y> = <X, LY> = \mathrm{II}(X, Y),$$

the second fundamental form of M relative to N. This follows from the fact that $V(X, Y) = -<LX, Y>N$ and V is symmetric; for

$$<V(X, Y), N> = <\overline{D}_X Y + D_X Y, N>$$
$$= <\overline{D}_X Y, N> = X<Y, N> - <Y, \overline{D}_X N> = -<Y, LX>$$

(see [2] for details). An umbilic on M is a point where $LX = kX$ for all X; that is, L is a multiple of the identity map. We define the *third fundamental form*, $\mathrm{III}(X, Y) = <LX, LY> = <L^2 X, Y>$, etc.

THEOREM 2. (Part of this theorem is well known.) *Let* M *be an* (n - 1)-*dimensional submanifold framed in* R^n *for which there exists a parallel submanifold* M_r $(r \neq 0)$, *and let* $f: M \to M_r$ *be the natural map. Then*

$$f_* X = X + r LX, \quad L_r f_* X = LX,$$

and f *preserves principal directions of curvature, umbilics, and the third fundamental form. Thus,*

$$<f_* X, f_* Y> = \mathrm{I}(X, Y) + 2r\,\mathrm{II}(X, Y) + r^2\,\mathrm{III}(X, Y),$$

where I, II, *and* III *are the first, second, and third fundamental forms on* M. *If* k *is a principal curvature of* M *at* m *in direction* X, *then* $k/(1 + rk)$ *is the corresponding principal curvature of* M_r *at* f(m) *in direction* $f_* X$.

Proof. To compute $f_* X$, we take a curve $\sigma(t) = (b_1(t), \cdots, b_n(t))$ with $X = \Sigma_1^n (db_i/dt)(0)e_i$, and then compute the tangent to $f \circ \sigma$ at $t = 0$. Let $N(\sigma(t)) = \Sigma a_i(t)e_i$; then

$$f \circ \sigma(t) = (\cdots, b_i(t) + ra_i(t), \cdots),$$

and its tangent at $t = 0$ is indeed $X + rL(X)$. Also,

$$N(\sigma(t)) = \sum a_i(t)e_i = N(f \circ \sigma(t))$$

from the definition of f; that is, we use the parallel translate of N to frame M_r. Thus

$$LX = \overline{D}_X N = \sum (da_i/dt)(0)e_i = \overline{D}_{f_* X} N = L_r f_* X .$$

This shows that

$$III_r(f_* X, f_* Y) = \langle L_r f_* X, L_r f_* Y\rangle = \langle LX, LY\rangle = III(X, Y) .$$

Now let X be a unit vector at m in M with $LX = kX$, so that $L_r(f_* X) = LX = kX$ and $f_* X = (1 + rk)X$. We show $1 + rk \neq 0$, since $1 + rk = 0$ implies that $f_* X = 0$, $L_r(f_* X) = kX = 0$; and thus, $k = 0$ so $1 = 0$. Consequently, $L_r(f_* X) = (k/1 + rk)f_* X$, showing that f preserves directions of curvature and umbilics. Finally, one can verify the expression for $\langle f_* X, f_* Y\rangle$ by direct computation using $f_* X = X + rLX$. Q. E. D.

Corollary. In the hypothesis of the above theorem, let $n = 3$; *and let the total curvature and mean curvature of* M (*and* M_r) *be denoted by* K (*and* K_r) *and* H (*and* H_r). *Then,*

$$K_r = K/(1 + rH + r^2 K) \quad \textit{and} \quad H_r = (H + 2rK)/(1 + rH + r^2 K) .$$

We next give a classical result (Theorem 3) and two analogous theorems (4 and 5). In all these theorems we will assume as a *standard hypothesis:* M is a complete connected 2-dimensional submanifold framed in R^3 for which there exists a parallel submanifold M_r $(r \neq 0)$, and $f: M \to M_r$ is the natural map.

We will continually make use of the classical theorem: A complete connected surface (2-dimensional submanifold) of umbilics in R^3 is a sphere or a plane. Let us denote by U the set of umbilics in a surface M, and let U^c be the set of non-umbilics. Since L is continuous, we know U is closed and U^c is open. The following theorem is standard, but we reprove it since its proof is a model for the following theorems.

THEOREM 3. *With the standard hypothesis, if* f *is conformal, then* M *is a sphere, plane, or has constant mean curvature* $H = -2/r$ *with no umbilics.*

Proof. Since f is conformal,

$$\langle f_* X, f_* Y\rangle = F\langle X, Y\rangle = \langle X + 2r LX + r^2 L^2 X, Y\rangle ;$$

hence,

$$r^2 L^2 + 2rL + (1 - F)I = 0 ,$$

where I is the identity map. But $L^2 - HL + KI = 0$ is the characteristic equation for L; hence

$$(H + 2/r)L = [K - (1 - F)/r^2] I .$$

If $H(m) + 2/r \neq 0$, then m is umbilic. For non-umbilical p in U^c, $H(p) = -2/r$ and $K(p) = (1 - F(p)/r^2$. Suppose that p_j is a sequence in U^c and $p_j \to m$ where m is in U. By continuity, $H(m) = -2/r$ and $K(m) = (1 - F(m))/r^2$. But also, $K(m) = 1/r^2$ since $k_i = -1/r$; and thus $F(m) = 0$, which is impossible since f conformal. Thus U^c is closed and open. Hence $M = U$ or $M = U^c$, and the only possibilities give the conclusion of the theorem. Q. E. D.

THEOREM 4. *With the standard hypothesis, if* f *is connexion preserving, then* M *is a sphere, a plane, or a right circular cylinder.*

Proof. Let X and Y be local vector fields on M. Since f is connexion preserving,

$$f_* D_X Y = D_X Y + r LD_X Y = D_{f_* X} f_* Y = D_{X+rLX} (Y + r LY).$$

Thus,

$$D_{LX} Y + D_X LY + rD_{LX} LY - LD_X Y = 0. \tag{1}$$

Let m be a non-umbilic point with $k_1 < k_2$ in the connected neighborhood A of m. Let X and Y be the orthonormal fields on A with $LX = k_1 X$, $LY = k_2 Y$.

We compute $D_X Y$ and $D_Y X$. Since

$$0 = X\langle Y, Y\rangle = 2\langle D_X Y, Y\rangle,$$

we know $D_X Y = aX$, and similarly $D_Y X = bY$. We show that $a = (Yk_1)/(k_2 - k_1)$ and $b = (Xk_2)/(k_1 - k_2)$. This follows since $D_X Y - D_Y X = [X, Y]$, and the Codazzi-Mainardi equation implies that $D_X LY - D_Y LX = L([X, Y])$. Thus

$$(Xk_2)Y + k_2 D_X Y - (Yk_1)X - k_1 D_Y X = a k_1 X - b k_2 Y,$$

and hence $(Xk_2) - k_1 b = - bk_2$ gives b, while $k_2 a - (Yk_1) = a k_1$ gives a.

Applying (1) to X and Y yields

$$k_1 aX + (Xk_2)Y + k_2 aX + rk_1(Xk_2)Y + rk_1 k_2 aX - k_1 aX = 0.$$

Thus $(Xk_2)(1 + rk_1) = 0$ and $k_2 a(1 + rk_1) = 0$. Therefore,

$$(Xk_2)(1 + rk_1) = 0 \tag{2}$$

$$k_2(Yk_1)(1 + rk_1) = 0. \tag{3}$$

Similarly, applying (1) to Y and X yields

$$(Yk_1)(1 + rk_2) = 0 \tag{4}$$

$$k_1(Xk_2)(1 + rk_2) = 0. \tag{5}$$

Applying (1) to X and X and using $D_X X = -aY$ yields

$$(Xk_1)(1 + rk_1) = 0 \tag{6}$$

$$(Yk_1)(2k_1 + rk_1^2 - k_2) = 0. \tag{7}$$

Similarly, applying (1) to Y and Y yields

$$(Yk_2)(1 + rk_2) = 0 \tag{8}$$

$$(Xk_2)(2k_2 + rk_2^2 - k_1) = 0. \tag{9}$$

From the proof of Theorem 2 we know that $1 + rk_i \neq 0$ (that is, f_* is always non-singular); thus (2), (4), (6) and (8) imply that $Xk_1 = Xk_2 = Yk_1 = Yk_2 = 0$ on A, and hence k_i is constant on A. Thus $a = 0$ and $b = 0$; so $D_X Y = 0$, $D_Y X = 0$, $D_X X = 0$, $D_Y Y = 0$ and $[X, Y] = 0$. The last condition implies that there exists a coordinate system in a neighborhood B of m with $B \subset A$, whose coordinate fields are X and Y; that is, B is then isometric to a piece of the plane. Hence k_1 and k_2 are constant on B, and $k_1 k_2 = K = 0$. Since $K \geq 0$ at an umbilic, $K \geq 0$ on all of M. Next we claim the set U^+ where $K > 0$ is an open and closed set of umbilics. It is clear that this set is open and a set of umbilics, since $K = 0$ at a non-umbilic. Therefore if $K(m) > 0$, m has a neighborhood of umbilics; then K is constant on this neighborhood and thus constant on the boundary of this neighborhood, which also consists of umbilics. Thus U^+ is closed. Hence either $M = U^+$ with $K > 0$ constant so M is a sphere, or $K = 0$ on M.

If $K = 0$ on M, then $H = 0$ at umbilical points, and $H \neq 0$ is constant on a neighborhood of a non-umbilical point. Thus the set of non-umbilics is open and closed, which implies that either M is umbilical and $K = 0$ so M is a plane, or M is non-umbilical with $K = 0$ and H constant so M is a right circular cylinder. Q. E. D.

THEOREM 5. *With the standard hypothesis, if* f *preserves the second fundamental form, then* M *is a plane.*

Proof. Inasmuch as $\langle LX, Y\rangle = \langle L_r f_* X, f_* Y\rangle = \langle LX, Y + rLY\rangle$, $rL^2 = 0$, or $L^2 = 0$. Thus, if X is a unit vector with $LX = kX$, then $L^2 X = k^2 X = 0$ and $k = 0$. Hence $L = 0$ or $\overline{D}_X N = 0$ for all X, and thus N is a constant vector on M so M is a plane. Q. E. D.

The following theorem should be well known.

THEOREM 6. *Let* M *be a complete connected* 2*-dimensional surface framed in* R^3 *with* I, II, *and* III *denoting its first, second, and third fundamental forms, respectively. If* I = II *or if* I = III, *then* M *is a sphere of radius* 1, *and conversely. If* II = III, *then* M *is a sphere of radius* 1, *a plane, or a right circular cylinder of radius* 1.

Proof. If I = II, then L is the identity map, k_i are both equal to 1, and all points are umbilical; so M is a sphere of radius 1 (and conversely). If I = III, then L^2 is the identity map. Let X be a unit vector with $LX = kX$, then $L^2 X = k^2 X = X$ so $k_i = 1$ or -1. Since L is continuous, k_i must be constant; and since a surface of constant negative curvature cannot be imbedded isometrically in R^3, the k_i must have the same sign and M is a unit sphere. If II = III, then $L^2 = L$, and it follows that $k_i = 0$ or 1. Thus if both $k_i = 1$, M is a unit sphere. If both $k_i = 0$, M is a plane. Finally, if $k_1 = 0$ and $k_2 = 1$, then M is a right circular cylinder. Q. E. D.

We last prove a well-known theorem since its proof fits the general pattern.

THEOREM 7. *Let* M *be a complete connected surface framed in* R^3, *and suppose its Gauss map* f *is conformal. Then* M *is a sphere or a minimal surface with strictly negative curvature and no umbilics. If, in addition, the Gauss map is connexion preserving, then* M *is a sphere.*

Proof. Since $\langle f_* X, f_* Y\rangle = \langle LX, LY\rangle = F\langle X, Y\rangle$,

$$L^2 X = FX, \quad L^2 - HL + KI = 0,$$

and consequently, $HL = (K + F)I$. If $H(m) \neq 0$, then m is umbilic and $K(m) > 0$ since $k_i \neq 0$. If m is umbilic and $H(m) = 0$, then $K(m) = -F(m) < 0$; but $K(m) = k^2 \geq 0$ at all umbilics. Thus the umbilical set is exactly the set of m where $H(m) \neq 0$, and hence this set is open and closed. If M is completely umbilical, then M is a sphere (since $F > 0$ eliminates the plane). If M has no umbilics, then $H = 0$ and $K = -F < 0$ on M. If f is connexion preserving, then F is constant and only the sphere case is possible. Q. E. D.

REFERENCES

1. L. P. Eisenhart, *An introduction to differential geometry*, Princeton University Press, Princeton, 1940.
2. N. Hicks, *Submanifolds of semi-Riemannian manifolds*, Rend. Cir. Mat. Palermo (to appear).
3. T. J. Willmore, *An introduction to differential geometry*, Oxford University Press, London, 1959.

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