KOSIŃSKI'S r-SPACES AND HOMOLOGY MANIFOLDS

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INTRODUCTION

A. Kosiński [3] defines a space X to be an r-space if (1) X is compact, metric, separable and finite dimensional, and (2) each point of X has an arbitrarily small closed neighborhood U such that Bd U is a (strong) deformation retract of U - y for each interior point y of U. He shows that r-spaces possess a very similar structure to that of topological manifolds, and that connected r-spaces which are finite polyhedra are actually integral homology manifolds.

The purpose of the present paper is to show that a connected r-space is a (locally orientable) singular homology manifold over an arbitrary coefficient field (THEOREM 1). This may be considered as an explanation of the fact that r-spaces resemble topological manifolds. I shall further show that a connected HLC r-space is a (locally orientable) singular homology manifold over any principal ideal domain K, and hence a Čech cohomology manifold over K (THEOREM 2 and the REMARK following it).

1. NOTATIONS AND PRELIMINARY REMARKS

Spaces under consideration are locally compact Hausdorff spaces. $H_c^q(X; \mathscr{F})$ will denote the sheaf theoretic q-th cohomology over the coefficient sheaf $\mathscr{F}$ with compact supports which is naturally isomorphic with the corresponding Čech cohomology. $H_q(E, F; K)$ will denote the usual relative singular q-th homology of the pair (E, F) over the coefficient ring K with compact supports. We shall omit the coefficient ring K from the homology and cohomology notations whenever it is clear from the context.

We shall review a few basic sheaf theoretic properties of the singular theory which will be used subsequently. Let S_q denote the K-module of finite singular q-th chains in X, and let $S = \Sigma S_q$. S may be regarded as a carapace with the obvious support function σ in the language of Cartan [1]. Furthermore S is homotopically fine and complete with respect to the family of compact supports, or equivalently $\Gamma_c(\mathscr{S}) \approx S$ where $\mathscr{S}$ is the sheaf induced by the carapace S. Let U be a subset of X, and let $S_U = \{a \in S \mid \sigma(a) \subset U\}$. If U is open, then $\Gamma_c(\mathscr{S} \mid U) \approx S_U$, and hence $H_q\Gamma_c(\mathscr{S} \mid U) \approx H_q(U)$ for each integer q. The derived sheaf $H_q(\mathscr{S})$, called the (local) singular homology sheaf over X can be readily seen to be generated by the presheaf: $V \to H_q(X, X - V)$. The stalk $H_q(\mathscr{S})_x$ is canonically isomorphic to $H_q(X, X - x)$.

DEFINITION. A space X is said to be a *(locally orientable) singular homology n-manifold over a principal ideal domain* K if

(1) $\dim_K X = n$ (the cohomology dimension),

(2) $H_q(X, X - x) = 0$ for $q \neq n$ and K for $q = n$, and

(3) the sheaf $H_*(\mathscr{S})$ is locally constant (local orientability condition).

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Remark. If a non-zero sheaf $\mathscr{F}$ over a locally connected space X is locally constant then the set $U = \{x \in X \mid F_x \neq 0\}$ is open and closed in X.

2. MAIN THEOREM

THEOREM 1. *Let* X *be a locally compact Hausdorff space, and let* K *be a field. Suppose that* X *satisfies the conditions:*

1. $\dim_K X$ *is finite.*

2. *Each point of* X *has a closed neighborhood* U *such that* Bd U *is a (strong) deformation retract of* U - y *for each interior point of* U. *Then each component of* X *is a singular homology* n*-manifold over* K *where* n *depends on the component.*

The proof for this will be formulated as a consequence of the following lemmas and a method used by Conner and Floyd.

LEMMA 1. *Let* X *be a locally compact Hausdorff space such that* $\dim_K X = n$ *(finite) and the singular homology sheaf* $H_*(\mathscr{S})$ *with the coefficients in* K *(a principal ideal domain) is locally constant. Then the least positive integer* m *for which* $H_m(\mathscr{S}) \neq 0$ *is not greater than* n.

Proof. Suppose that the conclusion of the lemma is false. Let V be an open set in X such that $H_*(\mathscr{S}) \mid V$ is constant. We shall consider the following exact sequence:

$$0 \to \mathscr{Z}_{n+1} \to \mathscr{S}_{n+1} \to \mathscr{S}_n \to \cdots \to \mathscr{S}_0 \to \mathscr{S}_{-1} \to 0,$$

where $\mathscr{S}_{q+1} \to \mathscr{S}_q$ is the boundary homomorphism, $\mathscr{Z}_{n+1} = \mathrm{Ker}(\mathscr{S}_{n+1} \to \mathscr{S}_n)$, $\mathscr{Z}_{n+1} \to \mathscr{S}_{n+1}$ is the inclusion map, $\mathscr{S}_{-1} = \mathrm{Coker}(\mathscr{S}_1 \to \mathscr{S}_0)$, and finally $\mathscr{S}_0 \to \mathscr{S}_{-1}$ is the projection map. For convenience, let $\mathscr{S}'^q = \mathscr{S}_{n+1-q} \mid V$, $0 \leq q \leq n+2$, and let $\mathscr{S}' = \Sigma \mathscr{S}'^q$. Then by a fundamental theorem on sheaf theory, there exists a spectral sequence $\{E_r^{p,q}\}$ such that

$$E_2^{p,q} \approx H_c^p(V; H^q(\mathscr{S}'))$$

and E_∞ is associated with a suitable filtration of $H^*(\Gamma_c(\mathscr{S}'))$. Since $H^q(\mathscr{S}') \approx 0$ for all $q \geq 1$ and $H^0(\mathscr{S}') \approx \mathscr{Z}_{n+1}$, by a standard argument on spectral sequences, we obtain the relations

$$H_c^p(V; \mathscr{Z}_{n+1}) \approx H^p(\Gamma_c(\mathscr{S}')) \approx H_{n+1-q}(V).$$

In particular, $H_c^{n+1}(V; \mathscr{Z}_{n+1}) \approx H_0(V) \not\approx 0$. This contradicts our hypothesis on the cohomology dimension of X, thus completing the proof.

Remark. Let X be a locally compact Hausdorff space such that the singular homology sheaf $H_q(\mathscr{S})$ is non-zero and locally constant for some q. Then X is locally arc-wise connected.

For, let U be an open set in X such that the sheaf $H_q(\mathscr{S})$ is constant over U and non-zero. Take an element a in S with $\sigma(\partial a) \subset X - U$, representing an element of $H_q(X, X - U)$ which is mapped onto a non-zero element of H (X, X - x) for all $x \in U$ under the homomorphism induced by $(X, X - U) \subset (X, X - x)$. Then $\sigma(a)$ contains U. a being a finite singular chain, $\sigma(a)$ is locally arc-wise connected; hence X is locally arc-wise connected.

LEMMA 2. *Let* X *be a locally compact Hausdorff space satisfying the conditions* 1-2 *of Theorem* 1, *where* K *is a principal ideal domain. Then the singular homology sheaf* $H_*(\mathscr{S})$ *is locally constant.*

Proof. Let U be a closed neighborhood of a point in X as in Condition 2 of Theorem 1, and let x be an interior point of U. By the definition of U, the inclusion map $\mathrm{Bd}\, U \subset U - x$ induces an isomorphism

$$H_q(\mathrm{Bd}\, U) \approx H_q(U - x)$$

for all q. It follows from a comparison of the homology sequences of the pairs (U, Bd U) and (U, U - x), and the "Five Lemma" that the inclusion map

$$(U, \mathrm{Bd}\, U) \subset (U, U - x)$$

induces an isomorphism

$$(*) \qquad H_q(U, \mathrm{Bd}\, U) \approx H_q(U, U - x)$$

for all q.

Let V be an open subset of X such that the closure of V is contained in the interior of U. Then the inclusion map $(U, U - V) \subset (X, X - V)$, being an excision, induces an isomorphism,

$$H_q(U, U - V) \approx H_q(X, X - V)$$

for all q. From this it follows that the sheaf $H_q(\mathscr{S}) | V$ is generated by the presheaf $W \to H_q(U, U - W)$, where W ranges over the family of all open subsets of V. Let us denote the constant sheaf $W \to H_q(U, \mathrm{Bd}\, U)$ by $H_q(U, \mathrm{Bd}\, U)$. Then there exists a presheaf homomorphism

$$f_W\colon H_q(U, \mathrm{Bd}\, U) \to H_q(U, U - W)$$

which is injective due to the isomorphism (*) above. Hence this induces an injective sheaf homomorphism

$$f\colon\ H_q(U, \mathrm{Bd}\, U) \to H_q(\mathscr{S}) | V.$$

Since the stalk at any $y \in V$ of the sheaf is nothing but $H_q(U, U - x)$, f is also surjective. As any sheaf homomorphism is an open map, f is a sheaf isomorphism. This completes the proof that $H_q(\mathscr{S}) | V$ is isomorphic with the constant sheaf $H_q(U, \mathrm{Bd}\, U)$ over V for all q and hence the sheaf $H_*(\mathscr{S})$ is locally constant.

Proof of Theorem 1. Let m and n be as in Lemma 1, and let V be as in the proof of Lemma 2 except that we further require that V is arc-wise connected. S being homotopically fine and complete with respect to the compact supports in X, there exists a spectral sequence $\{E_r^{p,q}\}$ (Cartan [1]) such that

$$E_2^{p,q} \approx H_c^q(V;\, H_q(\mathscr{S}) | V),$$

and $\Sigma p-q = -k$ $E_\infty^{p,-q}$ is associated with a suitable filtration of $H_k(V)$. We have already shown that $m \leq n$. Now we can show that $m \geq n$ by following the similar argument used by Conner and Floyd in the proof of their principal theorem (Theorem 6.1, [2]). This completes the proof.

3. HLC SPACES

Let K be a principal ideal domain. We recall that a space X is HLC over K if for any point x and any neighborhood U of x there is a neighborhood V of x contained in U such that the image of the homomorphism $\tilde{H}_*(V) \to \tilde{H}_*(U)$ induced by the inclusion map $V \subset U$ is zero, where $\tilde{H}_*$ denotes the augmented homology.

THEOREM 2. *Let* X *be a locally compact Hausdorff space that is* HLC *over a principal ideal domain* K. *If* X *satisfies the conditions* 1-2 *in Theorem* 1, *then each component of* X *is a singular homology* n-*manifold over* K *where* n *depends on the component.*

Undoubtedly, the following lemma is well known.

LEMMA 3. *Let* X *be a locally compact Hausdorff space that is* HLC *over a principal ideal domain* K. *If* (E, F) *and* (E', F') *are pairs of sub-spaces of* X *such that* $(\overline{E}, \overline{F})$ *are compact pairs contained in* (Int E', Int F') *then the image of the homomorphism*

$$H_q(E, F) \to H_q(E', F')$$

induced by the inclusion map $(E, F) \subset (E', F')$ *is finitely generated for each* q.

The absolute case of the lemma may be readily shown by a standard technique known for Čech theory which is also valid for the singular theory. The relative case follows from the absolute case and the following diagram:

$$\begin{array}{ccccc} H_q(E) & \to & H_q(E, F) & & \\ \downarrow & & \downarrow & & \\ H_q(E') & \to & H_q(E', F) & \to & H_{q-1}(F) \\ & & \downarrow & & \downarrow \\ & & H_q(E', F') & \to & H_{q-1}(F'), \end{array}$$

where the horizontal sequences are parts of singular homology sequences for pairs and the vertical homomorphisms are induced by inclusion maps. From the absolute case of the lemma, the images of the upper left and the lower right vertical homomorphisms are finitely generated. It is seen from the diagram that the image of the middle vertical homomorphism is also finitely generated.

LEMMA 4. *Let* X *be a locally compact Hausdorff space. Given an exact sequence* $0 \to M' \to M \to M'' \to 0$ *of* K-*modules, the induced sequence*

$$\cdots \to H_q(X, X - x; M) \to H_q(X, X - x; M'') \to H_{q-1}(X, X - x; M') \to \cdots$$

is exact for each $x \in X$. The proof is elementary.

Proof of Theorem 2. In terms of the notations of the proof of Lemma 2, the image of $H_q(U, \text{Bd } U)$ in $H_q(X, X - x)$ is a finitely generated module by Lemma 3. On the other hand, there are isomorphisms

$$H_q(U, \text{Bd } U) \approx H_q(U, U - x) \approx H_q(X, X - x)$$

which are induced by inclusion maps. Hence the module $H_q(X, X - x)$ is finitely generated. Now, we have arrived at a similar situation to that of Theorem 7.3 in [2] except that Conner and Floyd's case was in cohomology with coefficients in the integers. However, an analogous argument is also valid in our case of homology with coefficients in an arbitrary principal ideal domain. This completes the proof.

The author is indebted for a part of the proof above and the following remark to F. Raymond.

Remark. Let X be a locally compact Hausdorff space which is HLC over a principal ideal domain K. If X is a (locally orientable) singular homology n-manifold over K, then X is also a (locally orientable) Čech cohomology n-manifold over K.

Proof. Given $x \in X$ and an open neighborhood U of x, there is an open neighborhood V of x contained in U such that the inclusion map $V \subset U$ induces a trivial homomorphism $H_q(V) \to H_q(U)$ for all $q > 0$; hence by Alexander-Poincaré duality (Cartan [1]), the induced homomorphism $H_c^{n-q}(V) \to H_c^{n-q}(U)$ is also trivial for all $q > 0$. For $q = 0$, choose arc-wise connected V and U. Then the induced homomorphism $H_0(V) \to H_0(U)$ is an isomorphism. Again by the duality, $H_c^n(V) \to H_c^n(U)$ is an isomorphism, and each module is isomorphic with $H_0(V) \approx H_0(U) \approx K$.

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