COEFFICIENT ESTIMATES FOR STARLIKE MAPPINGS

Thomas H. MacGregor

Suppose that $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is regular and univalent for $|z| < 1$ and the image domain is starlike with respect to the origin. Then the Bieberbach conjecture $|a_n| \leq n$ holds, and equality occurs only for the functions $f(z) = z/(1 + \varepsilon z)^2$, where $|\varepsilon| = 1$ [4; p. 222].

The following generalization of this result was proved by Golusin [2]: *If*

$$f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1} \tag{1}$$

is regular, univalent, and starlike for $|z| < 1$, *then*

$$|a_{mk+1}| \leq \frac{1}{m!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2}{k}\right). \tag{2}$$

Equality in (2) occurs only for the functions $f(z) = z/(1 + \varepsilon z^k)^{2/k}$, where $|\varepsilon| = 1$. For $k = 1$ the estimate in (2) is the same as $|a_n| \leq n$. For $k = 2$ this theorem asserts that the coefficients of odd starlike functions satisfy the inequality $|a_n| \leq 1$. In [3] Golusin showed that the estimate $|a_n| \leq 1$ holds for $n = 3, 4, 5, \cdots$ if only $a_2 = 0$.

Theorem 1 in this paper implies that (2) holds if the hypothesis that f has the form (1) is replaced by the assumption that the power series for f begins with $z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \cdots$. This theorem also contains estimates for the coefficients which are not of the form a_{mk+1}. In particular, for $k = 2$, it gives upper bounds for $|a_n|$ which are less than 1 for each even value of n.

The proof of Theorem 1 depends essentially on a method introduced by Clunie [1]. In that paper the exact upper bounds are found for the coefficients of the functions $f(z) = 1/z + a_1 z + \cdots$ which are regular and univalent in $0 < |z| < 1$ and map $|z| < 1$ onto the complement of a point set starlike with respect to the origin.

THEOREM 1. *Suppose that* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is regular, univalent, and starlike for* $|z| < 1$. *Then*

$$|a_n| \leq \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2}{k}\right),$$

where $mk + 1 \leq n \leq (m+1)k$, $m = 1, 2, \cdots$.

Proof. f is univalent and starlike for $|z| < 1$ provided $\Re\{zf'(z)/f(z)\} > 0$. Let $g(z) = zf'(z)/f(z)$ and let

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$$h(z) = (g(z) - 1)/(g(z) + 1).$$

Then h is regular for $|z| < 1$, satisfies the inequality $|h(z)| < 1$, and has a power series which begins with $b_k z^k + b_{k+1} z^{k+1} + \cdots$. By equating coefficients of the power series on both sides of the equation

$$zf'(z) - f(z) = h(z)(zf'(z) + f(z)), \tag{3}$$

we obtain the relations

$$(n - 1)a_n = 2b_{n-1} \quad \text{for } n = k + 1, k + 2, \cdots, 2k. \tag{4}$$

Since $|h(z)| < 1$, it follows that $\Sigma_{n=k}^{\infty} |b_n|^2 \leq 1$, and therefore

$$\sum_{n=k}^{2k-1} |b_n|^2 \leq 1. \tag{5}$$

From (4) and (5) we find that

$$\sum_{n=k+1}^{2k} (n - 1)^2 |a_n|^2 \leq 4. \tag{6}$$

We rewrite (3) as follows:

$$\sum_{n=k+1}^{\infty} (n - 1)a_n z^n = h(z) \left\{ 2z + \sum_{n=k+1}^{\infty} (n + 1)a_n z^n \right\}$$

$$= h(z) \left\{ 2z + \sum_{n=k+1}^{p-k} (n + 1)a_n z^n \right\} + \sum_{n=p+1}^{\infty} c_n z^n.$$

This can also be written as

$$\sum_{n=k+1}^{p} (n - 1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = h(z) \left\{ 2z + \sum_{n=k+1}^{p-k} (n + 1)a_n z^n \right\}. \tag{7}$$

Since (7) has the form $F(z) = h(z)G(z)$, where $|h(z)| < 1$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2 \, d\theta \tag{8}$$

for each r $(0 < r < 1)$. Expressing (8) in terms of the coefficients in (7), we obtain the inequality

$$\sum_{n=k+1}^{p} (n - 1)^2 |a_n|^2 r^{2n} + \sum_{n=p+1}^{\infty} |d_n|^2 r^{2n} \leq 4r^2 + \sum_{n=k+1}^{p-k} (n + 1)^2 |a_n|^2 r^{2n}. \tag{9}$$

In particular (9) implies

$$\sum_{n=k+1}^{p} (n-1)^2|a_n|^2 r^{2n} \leq 4r^2 + \sum_{n=k+1}^{p-k} (n+1)^2|a_n|^2 r^{2n}. \tag{10}$$

By letting $r \to 1$ in (10), we conclude that

$$\sum_{n=k+1}^{p} (n-1)^2|a_n|^2 \leq 4 + \sum_{n=k+1}^{p-k} (n+1)^2|a_n|^2. \tag{11}$$

This inequality is equivalent to

$$\sum_{n=p-k+1}^{p} (n-1)^2|a_n|^2 \leq 4\left\{1 + \sum_{n=k+1}^{p-k} n|a_n|^2\right\}. \tag{12}$$

By an inductive argument we will establish the inequalities

$$\sum_{n=mk+1}^{(m+1)k} (n-1)^2|a_n|^2 \leq \left\{\frac{k}{(m-1)!}\prod_{\mu=0}^{m-1}\left(\mu+\frac{2}{k}\right)\right\}^2 \tag{13a}$$

$$\sum_{n=mk+1}^{(m+1)k} n|a_n|^2 \leq (mk+1)\left\{\frac{1}{m!}\prod_{\mu=0}^{m-1}\left(\mu+\frac{2}{k}\right)\right\}^2 \tag{13b}$$

for $m = 1, 2, 3, \cdots$.
For $m = 1$, (13a) is valid since it is the same as (6). We can prove (13b) for $m = 1$ by using (6) as follows.

$$\begin{aligned}\sum_{n=k+1}^{2k} n|a_n|^2 &= \frac{k+1}{k^2}\sum_{n=k+1}^{2k}\frac{k^2 n}{k+1}|a_n|^2 \\ &\leq \frac{k+1}{k^2}\sum_{n=k+1}^{2k}(n-1)^2|a_n|^2 \\ &\leq \frac{k+1}{k^2}\cdot 4 = (k+1)\left(\frac{2}{k}\right)^2.\end{aligned}$$

Now suppose that (13a) and (13b) hold for $m = 1, 2, \cdots, q-1$. Using (12) with $p = (q+1)k$ and the inductive hypothesis concerning (13a), we obtain the inequalities

$$\begin{aligned}\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2 &\leq 4\left\{1 + \sum_{n=k+1}^{qk} n|a_n|^2\right\} \\ &= 4\left\{1 + \sum_{m=1}^{q-1}\sum_{n=mk+1}^{(m+1)k} n|a_n|^2\right\}\end{aligned}$$

$$\leq 4\left\{1 + \sum_{m=1}^{q-1} (mk+1)\left[\frac{1}{m!}\prod_{\mu=0}^{m-1}\left(\mu + \frac{2}{k}\right)\right]^2\right\}$$

$$= \left\{\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu + \frac{2}{k}\right)\right\}^2.$$

The last equality can be readily proven with an inductive argument on q. This last sequence of inequalities implies (13a), where $m = q$.

Continuing our inductive argument, we use (13a) with $m = q$ to deduce (13b) for $m = q$ as follows.

$$\sum_{n=qk+1}^{(q+1)k} n|a_n|^2 = \frac{qk+1}{(qk)^2}\sum_{n=qk+1}^{(q+1)k} \frac{(qk)^2 n}{qk+1}|a_n|^2$$

$$\leq \frac{qk+1}{(qk)^2}\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2$$

$$\leq \frac{qk+1}{(qk)^2}\left\{\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left(\mu + \frac{2}{k}\right)\right\}^2$$

$$= (qk+1)\left\{\frac{1}{q!}\prod_{\mu=0}^{q-1}\left(\mu + \frac{2}{k}\right)\right\}^2.$$

This completes the proof of (13a) and (13b). The theorem follows from (13a).

It is not difficult to verify the following remarks. The estimate in Theorem 1 is precise if n is of the form $n = mk + 1$, and equality holds only for the functions

$$f(z) = z/(1 + \varepsilon z^k)^{2/k},$$

where $|\varepsilon| = 1$. If $k + 1 < n < 2k + 1$ the estimate for $|a_n|$ is exact for the same functions where k is replaced by n. For all other values of n, Theorem 1 does not give exact bounds.

COROLLARY. *If* $f(z) = z + \Sigma_{n=k+1}^{\infty} a_n z^n$ *is regular, univalent, and convex for* $|z| < 1$, *then*

$$|a_n| \leq \frac{k}{n(n-1)(m-1)!}\prod_{\mu=0}^{m-1}\left(\mu + \frac{2}{k}\right),$$

ere $mk + 1 \leq n \leq (m+1)k$, $m = 1, 2, \cdots$.

Proof. f is convex if and only if zf' is starlike.

Remarks. 1. If n is of the form $mk + 1$, then the estimate in Theorem 1 is the same as (2). For $k = 1$ this becomes $|a_n| \leq n$. For $k = 2$ Theorem 1 yields the bounds $|a_{2n+1}| \leq 1$, $|a_{2n+2}| \leq 2n/(2n+1)$, $n = 1, 2, 3, \cdots$. This improves $|a_n| \leq 1$ for each even n.

2. A generalization of the concept of starlike function is the concept of spiral-like function. Spiral-like functions are characterized by the condition

$$\mathfrak{R}\{\varepsilon z f'(z)/f(z)\} > 0$$

where $|\varepsilon| = 1$. If f(z) is spiral-like for $|z| < 1$, then f(z) is univalent for $|z| < 1$. Moreover, the coefficients of normalized spiral-like functions satisfy the inequality $|a_n| \leq n$ [5].

Theorem 1 remains valid if one replaces the condition that f be starlike by f is spiral-like. The proof is essentially the same. One considers

$$h(z) = (g(z) - \varepsilon)/(g(z) + \bar{\varepsilon})$$

where $g(z) = \varepsilon z f'(z)/f(z)$. This function h(z) satisfies the same conditions as h(z) in the proof of Theorem 1. (4) is replaced by

$$\varepsilon(n - 1)a_n = (\varepsilon + \bar{\varepsilon})b_{n-1} \text{ for } n = k + 1,\ k + 2, \cdots, 2k. \tag{14}$$

Nevertheless (6) remains valid, for it follows from (5) and (14) since $|\varepsilon| = 1$. (7) is replaced by

$$\sum_{n=k+1}^{p} \varepsilon(n - 1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = h(z)\left\{(\varepsilon + \bar{\varepsilon})z + \sum_{n=k+1}^{p-k} (n\varepsilon + \bar{\varepsilon})a_n z^n\right\}. \tag{15}$$

From (15) we obtain

$$\sum_{n=k+1}^{p} (n - 1)^2 |a_n|^2 \leq |\varepsilon + \bar{\varepsilon}|^2 + \sum_{n=k+1}^{p-k} |(n\varepsilon + \bar{\varepsilon})a_n|^2 . \tag{16}$$

Since (16) implies (11) we can establish Theorem 1 for the spiral-like functions by continuing exactly as we did in the proof of Theorem 1 for the starlike functions.

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Lafayette College