THE ALGEBRA OF SEMIPERIODIC SEQUENCES

I. David Berg

A sequence $z = \{z_n\}$ of complex numbers is called periodic if there exists positive m such that $z_{m+n} = z_n$ for all n. A sequence is called semiperiodic if it is in the uniform closure of the space of periodic sequences.

In a recent article [2], A. Wilansky and the author discussed the Banach space Q of semiperiodic sequences. There we mentioned that Q is not the space of almost-periodic functions on the integers.

In the present article, we put the obvious Banach algebra structure on Q and show that $Q = C(\bar{\omega})$, where $\bar{\omega}$ is the character group of R^0, where R^0 denotes the additive group of rationals mod 1 in the discrete topology. Hence, the theory of almost periodic functions on topological groups becomes available to us. The general problem of Bohr compactifications of locally compact abelian groups has been discussed by H. Anzai and S. Kakutani in [1], in which paper it is proved that $\bar{\omega}$ (there called the *universal monothetic Cantor group*) can be obtained as a Bohr compactification of the group of all integers.

In [2], A. Wilansky and the author showed that any matrix summing Q was bounded in the usual matrix norm. Here, we give a characterization of the matrices summing Q in terms of sequences of measures on $\bar{\omega}$.

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1. THE TOPOLOGICAL GROUP $\bar{\omega}$

Let (τ, ρ) denote the additive semi-group of positive integers τ with the metric ρ, where ρ is defined as follows:

$$\rho(x, y) = \frac{1}{n} \text{ if } n! \text{ divides } |x - y| \text{ and } (n + 1)! \text{ does not, } \rho(x, x) = 0.$$

Let $\bar{\omega}$ denote the completion of this metric space. It is easily verified that $\bar{\omega}$ is a compact topological group with metric ρ and addition + inherited from τ, for example, the identity 0 is the limit of the Cauchy sequence $\{n!\}$.

THEOREM 1. *Let* R^0 *denote the additive group of rationals* mod 1 *in the discrete topology. Then* $\bar{\omega}$ *is the character group of* R^0.

Proof. Let $x = \{x_n\} \in \bar{\omega}$. Then for $r \in R^0$, we define a homomorphism of R^0 into the circle group by

$$x(r) = \lim_{n \to \infty} \exp 2\pi i x_n r.$$

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To see that every character of R^0 is of this form, we observe that such a character χ is determined by its value on the set

$$\left\{\frac{1}{n!}\,\middle|\, n = 1, 2, \cdots\right\}$$

and that $\chi(1/m!)$ may be chosen to be any one of

$$\left\{\exp 2\pi i \frac{q_m}{m!}\,\middle|\, q_m \in [0, 1, \cdots, m! - 1] \text{ and } q_m \bmod (m-1)! = q_{m-1}\right\}.$$

We now note that for such a collection of q_m, $\{q_m\} \in \bar{\omega}$ and

$$\lim_{n \to \infty} \exp 2\pi i q_n \frac{1}{m!} = \exp 2\pi i q_m .$$

Hence we have generated all characters of R_0.

Finally, observing that for $x^1, x^2 \in \bar{\omega}$,

$$\left\{\rho(x^1, x^2) \leq \frac{1}{n}\right\} \Leftrightarrow \{x^1(r) = x^2(r) \text{ for all } r \in R^0 \text{ such that } n!\,r \equiv 0 \bmod 1\},$$

we see that the topology of $\bar{\omega}$ is indeed that of the character group of R^0. Q.E.D.

2. THE BANACH ALGEBRA Q

Define multiplication and involution in Q in the obvious pointwise manner. Then Q is clearly a function algebra.

THEOREM 2. $Q = C(\bar{\omega})$.

Proof. Any periodic sequence $P = \{p_n\}$ can be expressed uniquely as

$$p_n = \sum_{j=1}^{m_p} a_j \exp 2\pi i r_j n \qquad (r_j \in R^0).$$

Hence linear combinations of characters on $\bar{\omega}$ are dense in Q. Q.E.D.

We can now apply the entire theory of almost-periodic functions to Q. We are not interested in pursuing this further here and will merely note a few salient facts:

If $z = \{z_n\} \in Q$, the (Von Neumann) mean of Q is just the Cesàro mean given by

$$M(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} z_m .$$

Hence if we order $R^0 = \{r_m\}$, then for $z \in Q$,

$$z = \sum_{m=1}^{\infty} \exp i2\pi r_m \left[\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} z_j \exp(-i2\pi r_m j)\right]$$

where convergence is in the norm $|||z||| = (M(z\bar{z}))^{1/2}$.

We now characterize the space of matrices summing Q.

THEOREM 3. *Let* $A = (a_{ij})$ *be a matrix of complex numbers. Let* S *be an* $\bar{\omega}$ *disk. Define* $\mu_i(S)$ *by*

$$\mu_i(S) = \sum_{j \in S \cap \tau} a_{ij}.$$

Then A *sums* Q *if and only if*

1. A *is bounded.*
2. $\lim_{i \to \infty} \mu_i(S)$ *exists for each* $\bar{\omega}$ *disk* S.

Proof. Let A sum Q. We first note that A is bounded. For if $x = \{x_n\} \in \ell$,

$$\sup_{z \in Q, \|z\| = 1} \sum_{n=1}^{\infty} x_n z_n = \sum_{n=1}^{\infty} |x_n|.$$

Hence, by uniform boundedness, the statement $\Sigma_{n=1}^{\infty} x_n z_n$ exists for all $z \in Q$ implies $\{x_n\} \in \ell$. Hence, again by uniform boundedness,

$$\sup_i \left| \sum_{j=1}^{\infty} a_{ij} z_j \right| < \infty \Rightarrow \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

If S is an $\bar{\omega}$ disk, $S \cap \tau$ is a set of integers such that the characteristic function of $S \cap \tau$ is a periodic sequence of 1's and 0's. Call this sequence x. Then $\mu_i(S) = (Ax)_i$. Hence

$$\lim_{i \to \infty} \mu_i(S) = \lim_{i \to \infty} (Ax)_i.$$

Conversely, let the above hypotheses be satisfied. We observe that any periodic sequence is a finite sum of characteristic functions of $\bar{\omega}$ disks restricted to τ. Hence by the second hypothesis, A sums every periodic sequence. But since A is bounded, A sums Q.

REFERENCES

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Lehigh University
and
Yale University