SOME RADIUS OF CONVEXITY PROBLEMS

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1. INTRODUCTION

In a recent paper [2] the author obtained the following theorem which will be useful in certain applications in this note.

THEOREM 1. *If* $F(u, v)$ *is analytic in the* v*-plane and in the half-plane* $\Re u > 0$, *if* $P(z)$ *is regular with positive real part in* $\{|z| < 1\}$, *and if* $P(0) = 1$, *then on* $\{|z| = r < 1\}$

$$\min_{P} \min_{|z|=r} \Re F(P(z), zP'(z))$$

is attained only for a function $P = P_0$ *of the form*

$$P_0(z) = \frac{1+\alpha}{2}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right) + \frac{1-\alpha}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)$$

where $-1 \leq \alpha \leq 1$, $0 \leq \theta \leq 2\pi$.

The following corollary is easily verified.

COROLLARY 1. *The extremal function* P_0 *of Theorem* 1 *may be described by the equation*

$$\frac{P_0(z) - 1}{P_0(z) + 1} = \frac{bz - z^2}{1 - \bar{b}z},$$

where $b = \cos\theta + \alpha i \sin\theta$ *and* $-1 \leq \alpha \leq 1$.

It is well known [3] that if

$$f = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

maps the circle $\{|z| < 1\}$ onto a convex domain, then f is also starlike of order 1/2; that is,

$$\Re \frac{zf'(z)}{f(z)} \geq \frac{1}{2} \qquad (|z| < 1).$$

Conversely, if f is starlike of order 1/2 for $|z| < 1$, then it maps

$$\{|z| < (2(3)^{1/2} - 3)^{1/2} = 0.68\cdots\}$$

onto a convex domain, and the estimate is sharp. This result has been obtained just recently by T. MacGregor [1].

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One might ask a similar question in the meromorphic case. What is the radius of convexity for the class of functions

$$g = \frac{1}{z} + b_0 + b_1 z + \cdots + b_n z^n + \cdots \tag{1.1}$$

for which

$$\Re\left\{\frac{-zg'(z)}{g(z)}\right\} > \beta \geq 0 \qquad (|z| < 1) \tag{1.2}$$

for a given β $(0 < \beta < 1)$? If $\beta = 0$, the radius of convexity is $3^{-1/2}$. The author's proof [2] is long and cumbersome and a much neater proof will be described in this note, together with a proof of the theorem corresponding to the case $\beta = 1/2$ and a new proof of MacGregor's theorem [1]. All three theorems follow easily from the author's Theorem 1.

2. PROOFS OF THEOREMS

Let g be regular, univalent and starlike of order 1/2 for $0 < |z| < 1$ so that g is given by (1.1) and satisfies (1.2) with $\beta = 1/2$. We wish to find the largest circle with center at the origin within which every g satisfies the inequality

$$\Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \leq 0. \tag{2.1}$$

Since we may write

$$\frac{-zg'(z)}{g(z)} = \frac{1 + P(z)}{2} = \frac{1}{1 - w(z)}, \tag{2.2}$$

where $\Re P(z) > 0$ and $|w(z)| \leq |w| < 1$, it follows that (2.1) is equivalent to the inequality

$$\Re K(z) \geq 0, \quad K(z) = \frac{1 - zw'(z)}{1 - w(z)}. \tag{2.3}$$

In turn, the inequality (2.3) is equivalent to

$$\left|\frac{K(z) - 1}{K(z) + 1}\right| = \left|\frac{w(z) - zw'(z)}{2(1 - w(z)) + (w(z) - zw'(z))}\right| \leq 1. \tag{2.4}$$

This will be satisfied if

$$|w(z) - zw'(z)| \leq 1 - |w(z)|. \tag{2.5}$$

But since

$$w(z) = \frac{P(z) - 1}{P(z) + 1}$$

and

$$-\left\{1+\frac{zg''(z)}{g'(z)}\right\}=\frac{1}{2}(1+P(z))-\frac{zP'(z)}{1+P(z)},$$

we select $F(u, v)$ in Theorem 1 to be

$$F(u, v)=\frac{1}{2}(1+u)-\frac{v}{1+u}.$$

Corollary 1 then allows us to confine ourselves to extremal functions of the form

$$w=\frac{P(z)-1}{P(z)+1}=\frac{bz-z^2}{1-\bar{b}z} \qquad (|b|<1).$$

Hence $|w(z)| = |z|x$, where

$$x=\left|\frac{b-z}{1-\bar{b}z}\right|\leq 1.$$

We also see that

$$zw'(z)-w(z)=\frac{-(1-|b|^2)z^2}{(1-\bar{b}z)^2}$$

and

$$|zw'(z)-w(z)|=\frac{(|1-\bar{b}z|^2-|b-z|^2)r^2}{(1-r^2)|1-\bar{b}z|^2} \qquad (|z|=r)$$

(2.6)

$$=\frac{r^2-|w(z)|^2}{1-r^2}.$$

Consequently (2.5) becomes

$$\frac{r^2(1-x^2)}{1-r^2}\leq 1-rx,$$

or

$$0\leq 1-2r^2-(r-r^3)x+r^2x^2,$$

That is,

(2.7)
$$0\leq\left[rx-\frac{1}{2}(1-r^2)\right]^2+\frac{1}{4}(3-6r^2-r^4).$$

The last inequality is satisfied for $r\leq(2(3)^{1/2}-3)^{1/2}$. For the choice $z=i(2(3)^{1/2}-3)^{1/2}$, $b=i(2(3)^{1/2}/3-1)^{1/2}$, we find that

$$x=\left|\frac{b-z}{1-\bar{b}z}\right|=\frac{1-r^2}{2r},$$

so that equality holds in (2.7). Moreover, for this choice of z and b, $zw'(z) = 1$, which implies that $K(z) = 0$. It follows that the corresponding extremal function

$$g = \frac{1}{z}(1 + iz)^{\frac{1-\gamma}{2}} \cdot (1 - iz)^{\frac{1+\gamma}{2}}, \qquad \gamma = (2(3)^{1/2}/3 - 1)^{1/2},$$

has a radius of convexity equal to $(2(3)^{1/2} - 3)^{1/2}$. We have completed the proof of Theorem 2.

THEOREM 2. *Let*

$$g = \frac{1}{z} + b_0 + b_1 z + \cdots + b_n z^n + \cdots$$

be regular and starlike of order $1/2$ *for* $0 < |z| < 1$. *Then* g *maps*

$$\{ |z| < (2(3)^{1/2} - 3)^{1/2} \}$$

onto a domain the complement of which is convex. The estimate is sharp.

If $\beta = 0$ in (1.2) instead of $1/2$, a few modifications are needed in the proof of Theorem 2. The equalities (2.2) are now replaced by

$$\frac{-zg'(z)}{g(z)} = P(z) = \frac{1 + w(z)}{1 - w(z)},$$

and (2.3) and (2.4) now become

$$\Re\, K(z) \geq 0, \quad K(z) = \frac{(1 + w(z))^2 - 2zw'(z)}{1 - [w(z)]^2},$$

$$\left| \frac{K(z) - 1}{K(z) + 1} \right| = \left| \frac{w(z)^2 - [zw'(z) - w(z)]}{1 - [zw'(z) - w(z)]} \right| \leq 1. \tag{2.8}$$

Inequality (2.8) holds provided

$$|w(z)|^2 + 2|zw'(z) - w(z)| \leq 1. \tag{2.9}$$

Because of (2.6) we may write (2.9) in the form

$$|w(z)|^2 + \frac{2}{1 - r^2}(r^2 - |w(z)|^2) \leq 1,$$

or

$$(3r^2 - 1) - (1 + r^2)|w(z)|^2 \leq 0.$$

The last inequality is satisfied if $r \leq 3^{-1/2}$. We see that $K(z) = 0$ if $z = b = 3^{-1/2}i$ and the corresponding extremal function is

$$g = (1 + iz)^{1-c} \cdot (1 - iz)^{1+c},$$

where $c = 3^{-1/2}$.

THEOREM 3. *Let*

$$g = \frac{1}{z} + b_0 + b_1 z + \cdots + b_n z^n + \cdots$$

be regular and starlike for $\{0 < |z| < 1\}$. *Then* g *maps* $\{|z| \leq 3^{-1/2}\}$ *onto a domain the complement of which is convex. The estimate is sharp.*

If the function

$$f = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

is regular, univalent and starlike of order 1/2 for $|z| < 1$, we may let

$$\frac{zf'(z)}{f(z)} = \frac{1 + P(z)}{2} = \frac{1}{1 - w(z)}.$$

Then

$$K(z) = 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + zw'(z)}{1 - w(z)}$$

and

$$\left|\frac{K(z) - 1}{K(z) + 1}\right| = \left|\frac{[zw'(z) - w(z)] + 2w(z)}{[zw'(z) - w(z)] + 2}\right| \leq 1$$

provided

$$|zw'(z) - w(z)| + 2|w(z)| < 2 - |zw'(z) - w(z)|,$$

that is, provided

$$|w(z)| + |zw'(z) - w(z)| \leq 1. \tag{2.10}$$

Since (2.10) is precisely the same inequality as we encountered in (2.5), we again obtain (2.7). Hence (2.10) is satisfied for $|z| = r \leq (2(3)^{1/2} - 3)^{1/2}$.

For $z = (2(3)^{1/2} - 3)^{1/2}$ and $b = x = [(2(3)^{1/2} - 3)/3]^{1/2}$ we find that $1 + zw'(z) = 0$ and $K(z) = 0$. The corresponding extremal function is

$$f = z(1 - 2bz + z^2)^{-1/2}, \qquad b = \left(\frac{2(3)^{1/2} - 3}{3}\right)^{1/2}.$$

Thus MacGregor's theorem [1], which we state here as Theorem 4, follows as a consequence of Theorem 1.

THEOREM 4. *Let*

$$f = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be regular and starlike of order 1/2 *for* $|z| < 1$. *Then* f *is convex in* $\{|z| \leq (2(3)^{1/2} - 3)^{1/2}\}$. *The estimate is sharp.*

It would be of some interest to obtain the radius of convexity for functions starlike of arbitrary order β. Estimates may be obtained by the method used here. However, the problem of obtaining sharp estimates for β not 0 or 1/2 appears to be more difficult than for $\beta = 0$ or 1/2.

REFERENCES

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