INEQUALITIES FOR MONOTONIC ENTIRE FUNCTIONS

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1. Let $f(z)$ be an entire function of exponential type τ with $|f(x)| \leq 1$ for real x. Our object is to fill the gaps in the following table, where the entries under $|f'(x)|$ are due to S. N. Bernstein (for proofs see, for example, [1, Chapter 11] and [3]), and the others are consequences of a Phragmén-Lindelöf argument; see, for example [1, Chapter 6].

Hypothesis ↓	$\lvert f(x+iy)\rvert \leq$	$\lvert f'(x)\rvert \leq$	$\lvert f'(x+iy)\rvert \leq$
$\lvert f(x)\rvert \leq 1$	$e^{\tau\lvert y\rvert}$	τ	$\tau e^{\tau\lvert y\rvert}$
$\lvert f(x)\rvert \leq 1$ and $f(x)$ is monotone increasing		τ/π	

THEOREM 1. *If* $|f(x)| \leq 1$ *and* $f'(x) \geq 0$, *then*

$$|f'(x+iy)| \leq \frac{1}{2\pi}\left(\frac{\sinh \tau y}{y} + \tau\right). \tag{1.1}$$

The bound in (1.1) *is best possible (for each* y).

COROLLARY 1. *If* $f(x) \geq 0$ *and* $\int_{-\infty}^{\infty} f(x)dx \leq 1$, *then*

$$|f(x+iy)| \leq \frac{1}{4\pi}\left(\frac{\sinh \tau y}{y} + \tau\right); \tag{1.2}$$

in particular, $f(x) \leq \frac{1}{2}\tau/\pi$.

It is interesting to compare Corollary 1 with Korevaar's result [4] that if $\int_{-\infty}^{\infty} |f(x)|dx \leq 1$, then $|f(x+iy)| \leq (\pi y)^{-1} \sinh \tau y$, and in particular $|f(x)| \leq \tau/\pi$. Here the bound is not known to be best possible.

COROLLARY 2. *If* $f(x)$ *is real and* $\int_{-\infty}^{\infty} \{f(x)\}^2 dx \leq 1$, *then*

$$|f(x+iy)| \leq \frac{1}{4\pi}\left(\frac{\sinh 2\tau y}{y} + 2\tau\right).$$

THEOREM 2. *If* $|f(x)| \leq 1$ *and* $f'(x) \geq 0$, *then*

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(1.3) $$|f(x+iy)| \le \left(\frac{1}{2\pi} + o(1)\right)\frac{\sinh \tau y}{\tau y}, \quad |y| \to \infty .$$

The constant $(2\pi)^{-1}$ *is best possible.*

We may compare (1.3) with Duffin and Schaeffer's inequality (see [2]) $|f(x+iy)| \le \cosh \tau y$, which holds if $f(x)$ is real and $|f(x)| \le 1$. Not only does (1.3) say more for large $|y|$, but more for large τ: in other words, the larger the type τ, the more restrictive is the hypothesis that $f(x)$ is monotonic.

If we apply Bernstein's two theorems successively to f and $f' - \frac{1}{2}\tau/\pi$ under the hypotheses of Theorem 2, we find that $|f''(x)| \le \frac{1}{2}\tau^2/\pi$. Something is lost in this process, however, since the inequality is not sharp. We prove the following result.

THEOREM 3. *Under the hypotheses of Theorem* 2,

(1.4) $$|f''(x)| \le \frac{\tau^2}{2\pi\sqrt{3}} .$$

The constant is best possible.

2. The proofs of our theorems all depend on the analogue for entire functions of the Fejér-Riesz theorem on the representation of a nonnegative trigonometric polynomial as a square; see [1, p. 124]. We begin with Theorem 3, in which the details are the simplest.

Since $f'(x) \ge 0$ and $f(x)$ is bounded, $f'(x)$ is integrable. Hence

(2.1) $$f'(x) = |g(x)|^2 ,$$

where

(2.2) $$g(z) = \int_{-\tau/2}^{\tau/2} e^{izt} G(t)dt ,$$

and where

(2.3) $$\int_{-\tau/2}^{\tau/2} |G(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g(x)|^2 dx \le \frac{1}{\pi},$$

since

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} f'(x)dx = f(\infty) - f(-\infty) \le 2 .$$

From (2.1) and (2.2), it follows that

(2.4) $$f'(x) = \int_{-\tau/2}^{\tau/2}\int_{-\tau/2}^{\tau/2} e^{ix(t-u)} G(t)\overline{G}(u)dt\,du ,$$

and so

$$f''(x) = \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} e^{ix(t-u)}\, i(t - u)\, G(t)\, \overline{G}(u)\, dt\, du\,.$$

The maximum of

$$\left| \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} i(t - u)\, H(t)\, \overline{H}(u)\, dt\, du \right|$$

for

$$\left\{ \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} |H(t)|^2\, dt \right\}^{1/2} \leq 1$$

is the largest eigenvalue μ of the Hermitian kernel $i(t - u)$; hence, by (2.3), for each x, the maximum of $|f''(x)|$ under our conditions is $1/\pi$ times this number. This largest eigenvalue can be found either by calculating the first zero of the Fredholm determinant (μ is the reciprocal of this number) or by solving the integral equation with kernel $i(t - u)$. The calculations are elementary and will be omitted. The result is $\mu = \frac{1}{2}\tau^2/\sqrt{3}$; consequently

$$|f''(x)| \leq \frac{\tau^2}{2\pi\sqrt{3}}.$$

3. We next turn to Theorem 1. From (2.4), we see that

$$f'(x + iy) = \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} e^{ix(t-u)}\, e^{-y(t-u)}\, G(t)\overline{G}(u)\, dt\, du\,. \tag{3.1}$$

In this case the kernel is not Hermitian. However,

$$|f'(x + iy)| \leq \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} e^{-y(t-u)}\, |G(t)|\, |G(u)\, dt\, du$$

$$= \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} \cosh[y(t - u)]\, |G(t)|\, |G(u)|\, dt\, du$$

(to get the second form, write the first integral again with t and u interchanged, and add it to the first). The maximum of the last integral is now $1/\pi$ times the largest eigenvalue of the symmetric kernel $\cosh y(t - u)$.

Let us first observe that the value so obtained is actually an attained maximum for $|f'(x + iy)|$. There will exist for each y, a real function $H(t)$ such that

(3.2)
$$\int_{-\tau/2}^{\tau/2}\int_{-\tau/2}^{\tau/2} \cosh[y(t-u)]\, H(t)H(u)\,dt\,du$$

is maximized for $\int |H(t)|^2\,dt = 1/\pi$. We may assume $g(t)$ is nonnegative, since replacing $H(t)$ and $H(u)$ by $|H(t)|$ and $|H(u)|$, respectively, does not change $\int |H(t)|^2\,dt$ and cannot decrease (3.2). The function for which

(3.3)
$$f'(z) = \int_{-\tau/2}^{\tau/2}\int_{-\tau/2}^{\tau/2} e^{iz(t-u)}\, H(t)H(u)\,dt\,du$$

has the required form, and

$$f'(iy) = \int_{-\tau/2}^{\tau/2}\int_{-\tau/2}^{\tau/2} \cosh[y(t-u)]\, H(t)H(u)\,dt\,du = \mu/\pi .$$

In the present case it appears to be simplest to calculate μ by solving the integral equation

(3.4)
$$\mu\phi(u) = \int_{-\tau/2}^{\tau/2} \cosh[y(t-u)]\,\phi(t)\,dt\,du .$$

The details are elementary. The result is

(3.5)
$$\mu = \frac{\sinh \tau y}{2y} + \frac{1}{2}\tau .$$

Consequently,

$$|f'(x+iy)| \le \frac{1}{2\pi}\left(\frac{\sinh \tau y}{y} + \tau\right).$$

For $y = 0$, this reduces to Bernstein's second theorem.

4. The upper bound in Theorem 2 can be obtained from Theorem 1 by direct integration as follows:

(4.1)
$$f(x+iy) = f(x) + i\int_0^y f'(x+it)\,dt ,$$

(4.2)
$$\begin{aligned}|f(x+iy)| &\le 1 + \frac{1}{2\pi}\int_0^y \left(\frac{\sinh \tau t}{t} + \tau\right) dt \\ &= 1 + \frac{\tau y}{2\pi} + \frac{1}{2\pi}\int_0^y \frac{\sinh \tau t}{t}\,dt \sim \frac{1}{2\pi}\,\frac{\sinh \tau y}{y}, \qquad |y| \to \infty ,\end{aligned}$$

by L'Hôpital's rule.

To show that the constant in Theorem 2 is correct, we can use the following theorem of Widom [5] on the eigenvalues of rapidly increasing kernels.

WIDOM'S THEOREM. *Assume* $\gamma(x, y) = \log K(x, y)$ *is real and symmetric, belongs to* C^2 *and satisfies the conditions*

(i) $\gamma_y \geq 0$,

(ii) $\lim_{x,y\to\infty} y\gamma_y(x, y) = \infty$,

(iii) $\gamma_{yy} = o(\gamma_y^2)$ *and* $\gamma_{yx} = o(\gamma_y \gamma_x)$ *as* $x, y \to \infty$.

Then the largest eigenvalue of $K(x, y)$ *on* $(0, t)$ *is asymptotic to* $K(t, t)^2/K'(t, t)$ *as* $t \to \infty$. *The eigenvalue is simple and its corresponding eigenfunction is asymptotic to* $K(x, x)^{1/2}$. *All other eigenvalues are* $o(K(t, t)^2/K'(t, t))$.

In fact, from (4.1) and (2.4) the following estimate is obtained:

$$|f(x + iy)| \leq 1 + \left| \int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} e^{ix(t-u)}\, G(t)\overline{G}(u)\, \frac{1 - e^{-y(t-u)}}{t - u}\, dt\, du \right|.$$

Then, arguing as in Section 3, we see that the maximum of the double integral is, for a given y, the maximum of

$$\int_{-\tau/2}^{\tau/2} \int_{-\tau/2}^{\tau/2} \frac{\sinh y(t - u)}{t - u}\, |G(t)|\, |G(u)|\, dt\, du,$$

and so is $1/\pi$ times the largest eigenvalue of the integral equation with kernel $(t - u)^{-1} \sinh y(t - u)$, an equation to which Widom's Theorem is applicable.

However, the following procedure is more direct and more elementary. It is easily seen that the integral equation (3.4) has $\cosh yu$ as an eigenfunction. Hence, (3.2) is maximized by $H(t) = A \cosh yt$ where

$$A^2 = \frac{2}{\pi} \left(\frac{\sinh \tau y}{y} + \tau \right)^{-1}.$$

Consider the entire function f_y defined by (3.3) with this H. Then

$$f_y'(is) = A^2 \left(\frac{\sinh \frac{1}{2}\tau(y - s)}{y - s} + \frac{\sinh \frac{1}{2}\tau(y + s)}{y + s} \right)^2;$$

and, as $y \to \infty$,

$$i\,|f_y(iy) - f_y(o)| = \int_0^y f_y'(is)\, ds \sim A^2 \int_0^y \frac{\sinh^2 \frac{1}{2}\tau(y + s)}{(y + s)^2}\, ds$$

$$\sim \frac{2y}{\pi \sinh \tau y} \int_y^{2y} \frac{\sinh^2 \frac{1}{2}\tau t}{t^2}\, dt \sim \frac{\sinh \tau y}{2\pi \tau y},$$

by L'Hôpital's rule. Hence, (1.3) cannot hold with a smaller constant, since for each y there exists an f_y such that, as $y \to \infty$, (1.3) becomes an asymptotic equality.

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