SUMS OF NORMAL FUNCTIONS AND FATOU POINTS

Peter Lappan

Let C and D denote the unit circle and open disk, respectively. If f(z) is a complex valued function defined in D, the outer angular cluster set f(z) at a point $e^{i\theta}$ in C is denoted by $C_A(f, e^{i\theta})$ [6, p. 69], while the radial cluster set of f(z) at $e^{i\theta}$ is denoted by $C_R(f, e^{i\theta})$. The non-Euclidean hyperbolic distance between points z and z' in D is denoted by $\rho(z, z')$ [3, Chapter 2].

Bagemihl and Seidel have shown that every normal holomorphic function in D has a Fatou point [1, Theorem 4], and, in fact, that the set of Fatou points is dense on C [2, Corollary 1]. However, the author has shown that the sum of two normal holomorphic functions need not be normal [5, Theorem 4]. It is our present purpose to show that the sum of two normal holomorphic functions need not have a Fatou point.

We first prove a lemma concerning a Blaschke product.

LEMMA. *Let* E *be a prescribed countable set in* C. *Then there exists a Blaschke product* B(z) *such that*

(1) *for every* $e^{i\theta} \in E$, B(z) *has infinitely many zeros on the radius to* $e^{i\theta}$; *and*

(2) *there exist sequences* $\{R_n\}$ *and* $\{S_n\}$ *of real numbers, with* $0 < R_n < S_n < R_{n+1} < 1$, *such that* $|B(w_n)| \to 1$ *for every sequence* $\{w_n\}$ *with* $R_n < |w_n| < S_n$.

Proof. Let $a_n = 1 - 2^{-n}$ (n = 1, 2, $\cdots$); and let $\{e^{i\theta_n}\}$ be an enumeration of the elements of E, with every element of E appearing infinitely often in the enumeration.

We shall now locate the zeros $\{z_n\}$ of the Blaschke product. Set $z_1 = \frac{1}{2}e^{i\theta_1}$. Let R_1 be chosen with $|z_1| < R_1 < 1$ such that

$$\left|\frac{z - z_1}{1 - \bar{z}_1 z}\right| > a_1 \qquad (|z| > R_1).$$

Now choose S_1 with $R_1 < S_1 < 1$, and then choose $z_2 \in D$ such that $|z_2| > S_1$, $\arg z_2 = \theta_2$, and

$$\frac{z_2 - z}{1 - \bar{z}_2 z} > a_2 \qquad (|z| < S_1).$$

We now proceed inductively. Assume $z_1, z_2, \cdots, z_n$; $R_1, R_2, \cdots, R_{n-1}$; and $S_1, S_2, \cdots, S_{n-1}$ have been chosen such that

$$\arg z_j = \theta_j \qquad (1 \le j \le n), \tag{3}$$

$$|z_j| < R_j < S_j < |z_{j+1}| \qquad (1 \le j \le n - 1), \tag{4}$$

$$\left|\frac{z_{j+1} - z}{1 - \overline{z_{j+1}} z}\right| > a_{j+1} \qquad (|z| < S_j;\ 1 \le j \le n - 1), \tag{5}$$

Received November 9, 1962.

and

$$\prod_{k=1}^{j} \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| > a_j \qquad (|z| > R_j;\ 1 \leq j \leq n-1). \tag{6}$$

Choose R_n with $|z_n| < R_n < 1$ such that

$$\prod_{k=1}^{n} \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| > a_n$$

for $|z| > R_n$. Let S_n be any real number between R_n and 1; then choose z_{n+1} such that $\arg z_{n+1} = \theta_{n+1}$ and

$$\frac{|z_{n+1} - z|}{|1 - \bar{z}_{n+1} z|} > a_{n+1} \qquad (|z| < S_n).$$

Thus we can find sequences $\{z_j\}$, $\{R_j\}$, and $\{S_j\}$ satisfying (3), (4), (5), and (6) for all $j \geq 1$.

Condition (5) implies that

$$\prod_{k=j+1}^{\infty} \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| > \prod_{k=j+1}^{\infty} a_k \qquad (|z| < S_j). \tag{7}$$

Combining this with (6), we obtain the inequality

$$\prod_{n=1}^{\infty} \left| \frac{z_n - z}{1 - \bar{z}_n z} \right| > \prod_{k=j}^{\infty} a_k \qquad (R_j < |z| < S_j). \tag{8}$$

By (7), $\prod_{k=1}^{\infty} |z_k| > 0$, so that the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

converges. By the method of enumeration of E, (3) implies (1); and (8) implies (2) since $\prod_{k=j}^{\infty} a_k \to 1$ as $j \to \infty$. Thus the lemma is proved.

We note that this construction in no way confines $\rho(R_n, S_n)$. It will be useful in the following to choose R_n and S_n such that $\rho(R_n, S_n)$ is bounded away from zero.

We are now ready for the main result.

THEOREM. *There exists a holomorphic function* $f(z)$ *such that* $f(z)$ *is the sum of two normal functions, but* $f(z)$ *has no Fatou points.*

Proof. Let $\lambda(t)$ be the elliptic modular function with respect to the even modular group H (see [4, p. 157]), let $W(z) = i\,(1 - z)/(1 + z)$ (which maps D onto the upper half-plane), and let $u(z) = \lambda(W(z))$. Then the function

$$F(z) = u(z) + \frac{1}{u(z)} + \frac{1}{u(z) - 1}$$

is normal (as a rational function of a normal function) and is automorphic with respect to the group $G = W^{-1}HW$. If R is a specific fundamental region of this group, then every point of D is congruent to a point of $\overline{R}$, the closure of R (see Ford [4]).

Let E be the set of Fatou points of F(z). It is easily verified that E is a countable set, and that ∞ is the Fatou value at each point of E.

Let B(z) be a Blaschke product as described in the Lemma such that for every n, $\rho(z, z') \geq M_1 > 0$ for $|z| \geq S_n$, $|z'| \leq R_n$. We shall show that $f(z) = F(z) \cdot B(z)$ has no Fatou points, and that f(z) is the sum of two normal functions.

No point of E can be a Fatou point of f(z), for both 0 and ∞ are elements of $C_R(f, e^{i\theta})$ for $e^{i\theta} \in E$.

Now let $e^{i\theta} \notin E$ and let r_n be the hyperbolic midpoint of the line segment from R_n to S_n. If $\{f(r_n e^{i\theta})\}$ fails to have a limit, then $e^{i\theta}$ is not a Fatou point.

Suppose $f(r_n e^{i\theta}) \to \infty$. Then, since $e^{i\theta}$ is not a Fatou point of F(z), there exists a finite number α such that $\alpha \in C_A(F, e^{i\theta})$. However, since $|B(z)| < 1$ for all $z \in D$, there exists a number β with $|\beta| \leq |\alpha|$ such that $\beta \in C_A(f, e^{i\theta})$, and thus $e^{i\theta}$ is not a Fatou point of f(z).

Finally, suppose $f(r_n e^{i\theta}) \to \gamma$, where γ is finite. Then for each point $r_n e^{i\theta}$, there is a transformation T_n in G such that $z_n = T_n(r_n e^{i\theta})$ is in $\overline{R}$. The sequence $\{z_n\}$ has no limit point on C (since the four points of $\overline{R} \cap C$ are all Fatou points of F(z) with Fatou value ∞). We may select a subsequence $\{r_{n_k} e^{i\theta}\}$ such that $\{z_{n_k}\}$ converges to a point $\zeta \in D$; hence $F(r_{n_k} e^{i\theta}) \to F(\zeta)$. Let S be a non-Euclidean disk in D with center at ζ and radius $M_1/4$. There exists a point δ in S with $|F(\delta)| \neq |F(\zeta)|$, and there exists a sequence $\{w_k\}$ in D such that

$$\rho(w_k, r_{n_k} e^{i\theta}) < M_1/2$$

and $F(w_k) \to F(\delta)$. However, because of the distances involved, $R_{n_k} < |w_k| < S_{n_k}$ and hence, by (2), $|B(w_k)| \to 1$. Therefore, $|f(w_k)| \to |F(\delta)| \neq |\gamma|$, and, since $\{w_k\}$ approaches $e^{i\theta}$ angularly, $C_A(f, e^{i\theta})$ contains at least two elements and $e^{i\theta}$ is not a Fatou point of f(z).

We have now shown that f(z) has no Fatou points; it remains to be shown that f(z) is the sum of two normal functions. But clearly

$$f(z) = 2F(z) + (B(z) - 2) \cdot F(z),$$

where both 2F(z) and $(B(z) - 2) \cdot F(z)$ are normal; see [5, p. 188]. This completes the proof.

REFERENCES

1. F. Bagemihl and W. Seidel, *Behavior of meromorphic functions on boundary paths, with applications to normal functions*, Arch. Math. 11 (1960), 263-269.

2. ———, *Koebe arcs and Fatou points of normal functions*, Comment. Math. Helv. 36 (1961), 9-18.

3. C. Carathéodory, *Conformal representation*, 2d ed. Cambridge Tracts in Math. and Math. Phys., no. 28. Cambridge, at the University Press, 1952.

4. L. R. Ford, *Automorphic functions*, 2d ed., Chelsea Publ. Co., New York, 1951.

5. P. Lappan, *Non-normal sums and products of unbounded normal functions*, Michigan Math. Jour. 8 (1961), 187-192.

6. K. Noshiro, *Cluster sets*, Ergebnisse der Mathematik und ihrer Grenzgebiete, n.F., 28, Springer Verlag, Berlin · Göttingen · Heidelberg, 1960.

Lehigh University