TYPES OF AMBIGUOUS BEHAVIOR OF ANALYTIC FUNCTIONS, II

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Let f be a complex-valued function defined in the open disk D: $|z| < 1$. A point p in the unit circle C is an *ambiguous point* of f if there exist two arcs, each having p as one endpoint, but otherwise lying in D, such that on either of the arcs f(z) has a limit as z approaches p, but such that the limits are distinct. The concept is due to Bagemihl [1], as is the fundamental result that an arbitrary function can have only a countable number of ambiguous points. Lohwater and Piranian [2] have proved that, given an arbitrary countable set K on C, there is a meromorphic function f of bounded characteristic defined in D whose points of ambiguity constitute precisely K.

How far is it possible to prescribe the two asymptotic values at the points of ambiguity? Does the presence of many points of ambiguity guarantee the existence of asymptotic values at other boundary points? Can there exist many ambiguous points where there are more than two asymptotic values? In this note I give a family of examples relevant to these questions. References to other work on such questions can be found in [4].

THEOREM. *Let* W_1, W_2, W_3 *be three finite sets of points on the Riemann sphere. Then there exists a function* f, *meromorphic in* D, *such that* (1) *for each* j (j = 1, 2, 3) *the set of asymptotic values of* f *is* W_j, *at each point of a certain dense set* A_j *on* C, *and* (2) f *has asymptotic values only at points of* $A_1 \cup A_2 \cup A_3$. *The sets* A_j *do not depend on the sets* W_j.

Remark. It follows from Bagemihl's theorem that the sets A_j are necessarily countable.

Proof. I remark first that for each positive integer k there is an entire function ϕ_k having precisely k asymptotic values at infinity. For k = 1, we may take ϕ_1 to be z itself, and for k = 2, we may take ϕ_2 to be e^z. For $k > 2$, Valiron has shown [3, p. 140] that the function

$$\phi_k(z) = \int_0^z [\sin t^{(k-1)/2}]^2 t^{-k+1} dt$$

has the desired property. Note that $\phi_k(0) = 0$.

In the proof we shall also need a meromorphic function that has no asymptotic values at ∞. For this we choose any doubly periodic, meromorphic function ϕ_0 that is finite at 0 and 1.

Valiron shows that for $k > 2$ the $k - 1$ finite asymptotic values of ϕ_k are equally spaced on a circle around the origin. Thus $\phi_k(0)$ is not an asymptotic value of ϕ_k. For the other values of k, $\phi_0(0)$, $\phi_1(0)$, $\phi_2(0)$ are not asymptotic values of their

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functions. Next we remark that for each integer $k \geq 0$ and for any complex number α other than zero, the two functions $\phi_k(\alpha z)$ and $\phi_k(z)$ have the same sets of asymptotic values. Also given two finite sets, $\{z_1, z_2, \cdots, z_m\}$ and $\{w_1, w_2, \cdots, w_n\}$, of non-zero numbers, we can select α such that if z_j is any element of the first set and w_h is any element of the second set, then $\phi(\alpha z_j) \neq w_h$. Let the number of elements in W_1, W_2, W_3 be, respectively, r, s and t. There exist three numbers a, b, and c such that for all choices of α, β, and γ, (1) the function

$$\theta_1(z) = [\phi_r(\alpha/z) - a]^{-1}$$

has r asymptotic values at $z = 0$, all finite; (2) the function

$$\theta_2(z) = [\phi_s(\beta/(z - 1)) - b]^{-1}$$

has s asymptotic values at 1, all finite; (3) the function

$$\theta_3(z) = [\phi_t(\gamma z) - c]^{-1}$$

has t asymptotic values at infinity, all finite; and (4) none of the numbers $\theta_n(0)$, $\theta_n(1)$, $\theta_n(\infty)$ $(n = 1, 2, 3)$ that are well defined is an asymptotic value of any of the three functions. Each function is meromorphic over the entire sphere, except for one essential singularity. Now let

$$g(z) = c_1 \theta_1(z) + c_2 \theta_2(z) + c_3 \theta_3(z),$$

the numbers c_1, c_2, c_3 being nonzero. At each of the points $z = 0, 1, \infty$, two of the functions composing g are continuous and the third has asymptotic values. It is conceivable that an asymptotic value at one of these points may also be an asymptotic value at another. Appropriate choices of α, β, γ, c_1, c_2, c_3 will prevent this from happening. Thus, if an asymptotic value at $z = 0$ is also an asymptotic value at $z = 1$,

$$c_1 w_1 + c_2 \theta_2(0) + c_3 \theta_3(0) = c_1 \theta_1(1) + c_2 w_2 + c_3 \theta_3(1),$$

where for $n = 1, 2$, w_n is an asymptotic value of $\theta_n(z)$. This gives the relation

$$c_1(w_1 - \theta_1(1)) + c_2(\theta_2(0) - w_2) + c_3(\theta_3(0) - \theta_3(1)) = 0.$$

If either $w_1 \neq \theta_1(1)$ or $\theta_2(0) \neq w_2$, then the values of c_1, c_2, c_3 for which the equality holds form a thin subset of complex three-space. But we have already remarked that α, β, γ can be chosen so that $w_1 \neq \theta_1(1)$, $w_2 \neq \theta_2(0)$. More than that, α, β, γ can be chosen so that equality of any two asymptotic values will occur for only a thin set of values of c_1, c_2, c_3. Then the thinness of the c's guarantees the existence of one triple for which none of the undesired equalities of asymptotic values of g holds.

With these choices, we can now say that g has r asymptotic values at $z = 0$, s at $z = 1$, and t at $z = \infty$, all being different, and all finite. There is a rational function ρ that maps the asymptotic values at $z = 0$ onto the points in W_1, maps the asymptotic values at $z = 1$ onto the points in W_2, and maps the asymptotic values at $z = \infty$ onto W_3.

The composite function $\rho \circ g$ has each point of W_1 as an asymptotic value at $z = 0$, each point of W_2 as an asymptotic value at $z = 1$, and each point of W_3 as an asymptotic value at $z = \infty$; it has no other asymptotic values at these points, and has no asymptotic values at any other point.

Now let h be that elliptical modular function in D that has for vertices of its basic triangle the points 1, $\exp(\pi i/3)$, $\exp(2\pi i/3)$, and that maps these, respectively, onto 1, 0, and ∞. Let A_1 be the set of points on C where the radial limit of h(z) is 0, let A_2 be the set where the radial limit is 1, and let A_3 be the set where the radial limit is ∞.

Let P be a path approaching 0 on which $\rho \circ g$ has an asymptotic value, w. At each point of A_1 there is a path lying in D (except for an end point) whose image under the map h is the path P, and on this path $h \circ (\rho \circ g)$ has the asymptotic value w. Likewise, any asymptotic value of $h \circ (\rho \circ g)$ at a point of A_1 is an asymptotic value of $\rho \circ g$ at $z = 0$. Similar statements hold for A_2 and A_3. This completes the proof.

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