# NORMAL FUNCTIONS, THE MONTEL PROPERTY, AND INTERPOLATION IN $H^\infty$

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## 1. INTRODUCTION

The purpose of this paper is to exhibit an interrelationship among the subjects referred to in the title. As a by-product, we obtain new and somewhat shorter proofs of several known results.

## 2. SOME DEFINITIONS AND KNOWN THEOREMS

Let D denote the open unit disk, let K denote the family of one-to-one conformal mappings of D onto itself, and let f be a function which is meromorphic in D. Then f is said to be *normal* in D if the family $\{f \circ S: S \in K\}$ is normal in D in the sense of Montel. This class of functions was first introduced by Noshiro [9] in 1939; subsequently, it was rediscovered by Lehto and Virtanen [7], who remarked that the sum of a normal function and a bounded function (which is necessarily normal) is normal. Lappan [6] proved

THEOREM 2.1 (Lappan). *Corresponding to each unbounded normal holomorphic function* f *in* D, *there exist a Blaschke product* B *and a normal holomorphic function* g *in* D *such that* fB *and* f + g *are not normal in* D.

Throughout this paper, the Blaschke product

$$\prod_{1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

is denoted by $B(z; \{z_n\})$. (For a discussion of such products, see [11; p. 274].)

A function defined in D has the *Montel property* if the set of points on the unit circle C where the radial limit exists coincides with the set where the angular limit exists. In [4] the author proved

THEOREM 2.2. *Corresponding to each holomorphic function* f *in* D *having an infinite radial limit at the point* $\zeta$ *in* C, *there exists a Blaschke product* B *such that* fB *has an infinite radial limit at* $\zeta$ *but fails to have an angular limit there.*

A sequence $\{z_n\}$ of points in D is called an *interpolating sequence* if, given an arbitrary bounded sequence $\{w_n\}$, there exists a bounded holomorphic function f in D such that $f(z_n) = w_n$ $(n = 1, 2, \cdots)$. See [10], where further references are given. Carleson showed that a necessary and sufficient condition for $\{z_n\}$ to be an interpolating sequence is that

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(2.1) $$\inf_k \prod_{n \neq k} [z_k, z_n] > 0$$

where $[a, b] = |a - b|/|1 - \bar{a}b|$. Both Newman and Hayman proved that $\{z_n\}$ is an interpolating sequence provided $\sup_n (1 - |z_n|)/(1 - |z_{n-1}|) < 1$.

Throughout this paper, we denote the hyperbolic non-Euclidean distance between two points $z_1$ and $z_2$ in D by $\rho(z_1, z_2)$. (See [3; pp. 322-330] or [5; pp. 235-241].)

## 3. A UNIFYING PRINCIPLE

We observe that each of the above subjects involves Blaschke products. For example, (2.1) holds if and only if

$$\inf_k |B'(z_k; \{z_n\})|(1 - |z_k|^2) > 0.$$

The following theorem singles out a property of Blaschke products which serves as a unifying principle.

THEOREM 3.1. *Let* S *be a subset of* D *having at least one accumulation point on* C. *Then, corresponding to each positive number* $\gamma$, *there exists a denumerable subset* $\{z_n\}$ *of* S *such that the non-Euclidean disks* $\{z: \rho(z, z_n) < \gamma\}$ $(n = 1, 2, \cdots)$ *are disjoint and* $B(z; \{z_n\})$ *is bounded away from zero on the complement of their union, that is, on the set*

$$\bigcap_{n=1}^{\infty} \{z: \rho(z, z_n) \geq \gamma\}.$$

*Preliminary remarks.* That Theorem 3.1 is sharp follows from [2; Lemma 1]; conversely, Theorem 3.1 shows that Bagemihl and Seidel's lemma is the best possible. The reader should observe that Theorem 3.1 and the proof which we give below are implicitly contained in Newman's paper [8; pp. 504-505].

*Proof.* Set $c = e^{-4\gamma}$, and select a sequence $\{z_n\}$ of points from S such that $1 - |z_n| < c(1 - |z_{n-1}|)$ $(n = 2, 3, \cdots)$. If $m > n$, then

$$1 - |z_m| < c^{m-n}(1 - |z_n|), \quad \text{and} \quad |z_m| - |z_n| > (1 - c^{m-n})(1 - |z_n|).$$

From the inequality $|z_m| - |z_m||z_n| < 1 - |z_n|$, we conclude that

$$1 - |z_m||z_n| < (1 + c^{m-n})(1 - |z_n|)$$

and, hence, that

(3.1) $$\frac{|z_m| - |z_n|}{1 - |z_m||z_n|} > \frac{1 - c^{m-n}}{1 + c^{m-n}} \quad (m > n).$$

Next, consider any point z in the set $\bigcap\{z: \rho(z, z_n) \geq \gamma\}$. There exists an integer N such that $|z_N| \leq |z| < |z_{N+1}|$, provided $|z| \geq |z_1|$. For $n < N$,

(3.2) $$[z_n, z] \geq [|z|, |z_n|] \geq [|z_N|, |z_n|] > (1 - c^{N-n})/(1 + c^{N-n})$$

by [3; p. 321] and (3.1). Likewise, if $n > N + 1$,

$$[z_n, z] > (1 - c^{n-N-1})/(1 + c^{n-N-1}) . \tag{3.3}$$

Observing that the condition $\rho(z_n, z) \geq \gamma$ is equivalent to $[z_n, z] \geq \tanh \gamma$, we conclude from (3.2) and (3.3) that

$$|B(z; \{z_n\})| = \prod_{n<N} [z_n, z] \cdot [z_N, z] \cdot [z_{N+1}, z] \cdot \prod_{n>N+1} [z_n, z]$$

$$> [\tanh \gamma \prod_1^\infty \{(1 - c^k)/(1 + c^k)\}]^2 .$$

We have assumed that $|z| \geq |z_1|$ and that $N > 1$; the remaining cases are left to the reader.

Finally, let us prove that the disks are disjoint. From (3.1) we see that, for $m > n$,

$$[z_m, z_n] > (1 - c^{m-n})/(1 + c^{m-n}) \geq (1 - c)/(1 + c) .$$

Hence,

$$\rho(z_m, z_n) > \tanh^{-1}\{(1 - c)/(1 + c)\} = 2\gamma .$$

The disjointness now follows from the triangle inequality, and this completes the proof.

## 4. SOME CONSEQUENCES

Theorem 3.1 is obviously connected with the interpolation problem mentioned above; see [8; pp. 504-505]. Indeed, every set of points in D having at least one accumulation point on C contains a countable subset which constitutes an interpolating sequence. To see this, choose $\{z_n\}$ as in Theorem 3.1, and let $B_k$ be the Blaschke product whose zeros are $z_1, \cdots, z_{k-1}, z_{k+1}, \cdots$. Then, for some positive number $\alpha$, $|B_k(z)| \geq |B(z; \{z_n\})| > \alpha$ for all z satisfying the equation $\rho(z, z_k) = \gamma$; and by the minimum modulus principle, $|B_k(z_k)| > \alpha$ $(k = 1, 2, \cdots)$, so that condition (2.1) holds for the sequence $\{z_n\}$.

Next, let us prove the first part of Theorem 2.1. Let f be an unbounded normal holomorphic function in D. Select a sequence $\{\xi_n\}$ of points in D such that $f(\xi_n) \to \infty$ as $n \to \infty$. For each positive integer n, take $z_n$ to be any point for which $\rho(z_n, \xi_n) = 1$, and observe that $|z_n| \to 1$ as $n \to \infty$. Using Theorem 3.1, select a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $B(z; \{z_{n_k}\})$ is bounded away from zero on the set

$$\bigcup_{k=1}^\infty \{z: \rho(z, z_{n_k}) = 1\} .$$

Let $h(z) = f(z)B(z; \{z_{n_k}\})$. Then $h(z_{n_k}) = 0$ $(k = 1, 2, \cdots)$ and $h(\xi_{n_k}) \to \infty$ as $k \to \infty$.

Using Lemma 3 (or Lemma 4) of [6], we conclude that h is not normal in D.

As a third application of Theorem 3.1, let us give a new proof of Theorem 2.2. Let the set S in the statement of Theorem 3.1 be a non-radial line segment contained in D and having one endpoint at $\zeta$. Assume for the moment that the hyperbolic non-Euclidean distance between the radius to $\zeta$ and the set S is positive, that is, that

$$\inf\{\rho(z, r\zeta): z \in S;\ 0 \leq r < 1\} > 0. \tag{4.1}$$

Take $\gamma$ to be this distance, and select $\{z_n\}$ as in Theorem 3.1. Then $\inf\{|B(r\zeta; \{z_n\})|: 0 \leq r < 1\} > 0$, and $f(z)B(z; \{z_n\})$ has the desired properties. To see that (4.1) holds, let $\theta$ $(0 < \theta < \pi/2)$ be the angle between the radius in question and the segment S. Then, if a is in S, and if b is on the radius, a simple geometric argument yields the conclusion

$$|a - b| \geq |\zeta - a|\sin\theta > (1 - |a|)\sin\theta = 2\alpha(1 - |a|)/(1 - \alpha),$$

where $\alpha = \sin\theta/(2 + \sin\theta)$. This, in turn, implies that $[a, b] > \alpha$, that is, that $\rho(a, b) > \tanh^{-1}\alpha$; for, otherwise,

$$\begin{aligned} |a - b| &= (1 - |a|^2)[a, b]\,|1 - \bar{a}(a - b)/(1 - \bar{a}b)|^{-1} \\ &\leq \alpha(1 - |a|^2)/(1 - |a|\alpha) \\ &< 2\alpha(1 - |a|)/(1 - \alpha), \end{aligned}$$

which is a contradiction.

As an analogue of Lappan's theorem, we prove the following result, which is a slight extension of a previous theorem [4].

THEOREM 4.1. *Corresponding to each normal holomorphic function* f *in* D *approaching* $\infty$ *along some boundary path, there exist* (a) *a Blaschke product* B *and* (b) *a normal holomorphic function* g *in* D *such that* fB *and* f + g *do not have the Montel property.*

*Proof.* Let f be the function described in the theorem. By [1; Corollary 1], the path along which f(z) approaches $\infty$ must terminate at some point of C; and then, by [7; Theorem 2], we conclude that f(z) has $\infty$ as an angular limit at that boundary point. Part (a) of the theorem now follows from Theorem 2.2; and part (b) follows by [6; Lemma 2].

## 5. ANOTHER TECHNIQUE

Let us conclude by proving a slightly weakened version of Theorem 3.1, which suffices for most of the applications in the preceding section and, moreover, which illustrates a technique that turns out to be useful in other connections. The reader should have no trouble in using the ideas of this section to give a new and shortened proof of Theorem 3.1. Under the hypothesis of Theorem 3.1, we shall show the existence of a denumerable subset $\{z_n\}$ of S such that the disks $\{z: \rho(z, z_n) \leq \gamma\}$ are disjoint and $B(z; \{z_n\})$ is bounded away from zero on $\bigcup\{z: \rho(z, z_n) = \gamma\}$.

The proof proceeds as follows. Letting $H_z = \{z': \rho(z, z') = \gamma\}$, one can easily verify that

(5.1) $$r_z = \sup\{|z'|: z' \in H_z\} < 1,$$

(5.2) $$0 < 1 - [a, z] \leq \left|1 - \frac{|a|}{a}\frac{a - z}{1 - \bar{a}z}\right| \leq \frac{(1 + |z|)(1 - |a|)}{1 - |z|}$$

for $|z| < 1$ and $0 < |a| < 1$, and

(5.3) $$\inf\{|z'|: z' \in H_z\} \to 1 \quad \text{as } |z| \to 1.$$

Let $c_n = 1 - 2^{-n}$ $(n = 1, 2, \cdots)$. Then choose $z_1$ to be any point in $S - \{0\}$, and select $z_2$ in $S$ in such a way that

(5.4) $$[z_2, z] > c_2 \quad \text{for all } z \text{ in } H_{z_1}$$

and

(5.5) $$[z_1, z] > c_1 \quad \text{for all } z \text{ in } H_{z_2}.$$

In view of (5.1) and (5.2), (5.4) holds provided

$$(1 + r_{z_1})(1 - r_{z_1})^{-1}(1 - |z_2|) < \frac{1}{2}(1 - c_2);$$

and (5.5) holds for $|z_2|$ sufficiently close to 1 in virtue of (5.3) and the fact that $[z_1, z] \to 1$ uniformly as $|z| \to 1$. Continuing this process by induction, one can select a sequence $\{z_n\}$ in such a way that $[z_n, z] > c_n$ for all $z$ in $H_{z_j}$ provided $j \neq n$. Doing this, we see that

$$\prod_1^\infty [z_n, z] > c_1 c_2 \cdots c_{m-1}(\tanh\gamma)c_{m+1} \cdots$$

for all $z$ in $H_{z_m}$ $(m = 1, 2, \cdots)$, which completes the proof.

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