RELATIONSHIPS AMONG THE SOLUTIONS OF TWO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Given two systems of ordinary differential equations,

$$\dot{x} = A(t)x + g(t, x) \qquad \left(\cdot = \frac{d}{dt}\right), \tag{1}$$

$$\dot{y} = A(t)y, \tag{2}$$

the following problem is posed:

If $y(t)$ $\{x(t)\}$ is a solution of (2) $\{(1)\}$, is there a solution $x(t)$ $\{y(t)\}$ of (1) $\{(2)\}$ such that $x(t) - y(t) \to 0$ as $t \to \infty$?

In this work we use a topological method of Ważewski to discuss this problem. Reference to the above problem can be found in a book by L. Cesari [1, Section 3.7, p. 41 and Section 3.9.xi, p. 47].

We are going to state here two theorems of Ważewski used in this paper, giving first some definitions and notations.

Hypothesis H. (a) *The real-valued functions* f_i $(i = 1, \cdots, n)$ *of the real variables* $t, x_1, \cdots, x_n$, *are continuous in a set* $\Omega \subset R^{n+1}$.

(b) *Through every point of* Ω *passes only one integral curve of the system*

$$\dot{x} = f(t, x), \tag{3}$$

where

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} f_1(t, x_1, \cdots, x_n) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t, x_1, \cdots, x_n) \end{pmatrix},$$

and $(t, x) \in \Omega$.

Let ω and Ω be open sets of R^{n+1} with $\omega \subset \Omega$, and denote by $B(\omega, \Omega)$ the boundary of ω in Ω.

Let $P_0 = (t_0, x_0) \in \Omega$. We write $I(t, P_0) = (t, x(t, P_0))$, where $x(t, P_0)$ is the integral curve of the system (3) passing through the point P_0.

Let $(\alpha(P_0), \beta(P_0))$ be the maximal open interval in which the integral curve passing through P_0 exists. We write

$$I(\triangle, P_0) = \{(t, x(t, P_0)) \mid t \in \triangle\}$$

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for every $\triangle$ contained in $(\alpha(P_0), \beta(P_0))$.

The point $P_0 = (t_0, x_0) \in B(\omega, \Omega)$ is a *point of egress* from ω (with respect to the system (3) and the set Ω) if there exists a positive number δ such that $I([t_0 - \delta, t_0), P_0) \subset \omega$; P_0 is a *point of strict egress* from ω if P_0 is a point of egress and if there exists a positive number δ such that $I((t_0, t_0 + \delta], P_0) \subset \Omega - \bar{\omega}$. The set of all points of egress (strict egress) is denoted by S (S^*).

If A and B are any two sets of a topological space with $A \subset B$ and if $K: B \to A$ is a continuous mapping from B onto A such that $K(P) = P$ for every $P \in A$, then K is a *retraction* from B into A, and A is a *retract* of B.

WAZEWSKI'S FIRST THEOREM. *Suppose that the system* (3) *and the open sets* $\omega \subset \Omega \subset R^{n+1}$ *satisfy the following hypotheses:*

1) *Hypothesis* H

2) $S = S^*$.

3) *There exists a set* $Z \subset \omega \cup S$ *such that* $Z \cap S$ *is a retract of* S, *but it is not a retract of* Z.

Then there exists at least one point $P_0 = (t_0, x_0) \in Z - S$ *such that* $I(t, P_0) \subset \omega$ *for every* t $(t_0 \leq t < \beta(P_0))$.

Ważewski's theorem [4, Theorem 1, p. 299] is actually more general than the one stated above.

If $f_i(t, x_1, \cdots, x_n)$ $(i = 1, \cdots, n)$ are complex-valued functions of the real variable t and of the complex variables $x_1, \cdots, x_n$, the n-dimensional complex system (3) can be considered as a 2n-dimensional real system, so that the theorem of Ważewski is also extensible, in a natural way, to complex systems [3, p. 19, Section 1 and p. 21, Section 2].

Let $g(t, x_1, \cdots, x_n) = g(t, x)$ be a real-valued function belonging to C^1 on a set $\Omega \subset R^{n+1}$, that is, suppose all first partial derivatives of g exist and are continuous on Ω.

Let $P_0 = (t_0, x_0) \in \Omega$, and let $x(t)$ be the integral curve of the system (3) passing through the point P_0. We set $\phi(t) = g(t, x(t))$. The derivative of $g(t, x)$ at the point $P_0 = (t_0, x_0)$, with respect to the system (3) is by definition $\dot{\phi}(t_0)$ and is denoted by $[D_{(3)}\, g(P)]_{P_0}$.

Regular polyfacial set. Let $\ell^i(t, x)$ and $m^j(t, x)$ $(i = 1, \cdots, p;\ j = 1, \cdots, q)$ be real-valued functions belonging to C^1 on an open set $\Omega \subset R^{n+1}$.

Let

$$\omega = \{P \in \Omega \mid \ell^i(P) < 0,\ i = 1, \cdots, p,\ m^j(P) < 0,\ j = 1, \cdots, q\},$$

$$L^i = \{P \in \Omega \mid \ell^i(P) = 0,\ \ell^k(P) \leq 0,\ m^j(P) \leq 0,\quad k = 1, \cdots, p;\ j = 1, \cdots, q\},$$

$$M^j = \{P \in \Omega \mid m^j(P) = 0,\ \ell^i(P) \leq 0,\ m^k(P) \leq 0,\quad i = 1, \cdots, p;\ k = 1, \cdots, q\}.$$

Suppose that for all i, j $(1 \leq i \leq p;\ 1 \leq j \leq q)$, $[D_{(3)}\, \ell^i(P)]_{P \in L^i}$ is positive, and $[D_{(3)}\, m^j(P)]_{P \in M^j}$ is negative.

Under these hypotheses, the set ω is called a *regular polyfacial set.* The L^i are called positive faces, and the M^j are called negative faces of ω.

WAŻEWSKI'S SECOND THEOREM. *Let* Ω *be an open set in* R^{n+1} *where the system* (3) *satisfies the hypothesis* H.

Let $\omega \subset \Omega$ *be a regular polyfacial set.*

Then $S = S^* = \bigcup_{i=1}^{p} L^i - \bigcup_{j=1}^{q} M^j$ [4, Theorem 5, p. 310].

For convenience we shall write the systems (1) and (2) in the following way:

$$\dot{x}_i = \sum_{j=1}^{n} f_{ij}(t)x_j + g_i(t, x), \tag{1}$$

$$\dot{y}_i = \sum_{j=1}^{n} f_{ij}(t)y_j. \tag{2}$$

2. A THEOREM ON SOLUTIONS DEFINED IN THE FUTURE

In the sequel it is always supposed that the systems (1) and (2) satisfy the hypothesis H in $[T, \infty) \times \Gamma$, where T is a real number and Γ is an open set in $\{x \mid \|x\| < \infty\}$. We denote the real part of a complex-valued function $f(t)$ by $\Re(f(t))$. If $z = z(t)$ is a complex-valued n-vector, $t_0 \geq T$, and $\varepsilon > 0$, we define

$$W_{\varepsilon, t_0, z} = \bigcup_{t \geq t_0} \{t\} \times V_{\varepsilon(z(t))},$$

where

$$V_{\varepsilon, z(t)} = \{x \mid \|x - z(t)\| < \varepsilon\}.$$

We say that a solution $x(t)$ is defined in the future if the maximum open interval in which it is defined contains some half-line $[\tau, \infty)$. A solution $x(t)$ defined in the future is said to be bounded in the future if it is defined and bounded in some half-line $[\tau, \infty)$.

THEOREM 1. *Suppose* $y = y(t)$ *is a given solution of* (2) *and there exist an* $\varepsilon > 0$ *and a* $t_0 \geq T$ *such that*

$$W_{\varepsilon, t_0, y} \subset \Omega = (T, \infty) \times \Gamma.$$

If there exist continuous functions $h_j(t)$ *such that* $|g_j(t, x)| \leq h_j(t)$ *for all* $(t, x) \in W_{\varepsilon, t_0, y}$ $(j = 1, \cdots, n)$, *and if*

$$\int_t^{\infty} h_k(v)\left[\exp \int_v^t \Re(f_{kk}(s))\, ds\right] dv \to 0 \quad \text{as} \quad t \to \infty \quad (k = 1, \cdots, n), \tag{4}$$

$$\int_t^{\infty} |f_{ij}(v)|\left[\exp \int_v^t \Re(f_{ii}(s))\, ds\right] dv \to 0 \quad \text{as } t \to \infty \quad (i \neq j). \tag{5}$$

Then there exists a solution x(t) *of* (2) *defined in the future such that*

$$x(t) - y(t) \to 0 \text{ as } t \to \infty .$$

Proof. We define $\omega = \{P \in \Omega \mid |x_i - y_i(t)| < \phi_i(t),\ t \geq t_1 \geq t_0\}$, where the function $\phi_i(t)$ and the constant t_1 will be chosen so that for all $t \geq t_1$ $(i = 1, \cdots, n)$, $\phi_i(t) > 0$, the ϕ_i are differentiable, $\lim_{t\to\infty} \phi_i(t) = 0$, and ω a regular polyfacial set.

If we put

$$\ell^i(P) = |x_i - y_i(t)|^2 - \phi_i^2(t) \qquad (i = 1, \cdots, n),$$
$$m^1(P) = t_1 - t,$$

then $\omega = \{P \in \Omega \mid \ell^i(P) < 0,\ i = 1, \cdots, n,\ m^1(P) < 0\}$.

For all i $(1 \leq i \leq n)$,

$$L^i = \{P \in \Omega \mid |x_i - y_i(t)| = \phi_i(t),\ |x_j - y_j(t)| \leq \phi_j(t),\ j = 1, \cdots, n,\ t \geq t_1\},$$
$$M^1 = \{P \in \Omega \mid |x_i - y_i(t)| \leq \phi_i(t),\ t = t_1\}.$$

An easy computation shows that

$$\frac{1}{2}[D_{(1)}\ell^i(P)]_{P\in L^i} \geq |x_i - y_i(t)|^2 \Re(f_{ii}(t)) - \sum_{j\neq i} |f_{ij}(t)| \cdot |x_i - y_i(t)| \cdot |x_j - y_j(t)|$$
$$- |g_i(t, x)| \cdot |x_i - y_i(t)| - \phi_i(t)\dot{\phi}_i(t)$$
$$\geq \phi_i^2(t)\Re(f_{ii}(t)) - \sum_{j\neq i} |f_{ij}(t)|\phi_i(t)\phi_j(t) - |g_i(t, x)|\phi_i(t) - \phi_i(t)\dot{\phi}_i(t).$$

As we want $\phi_i(t)$ to be positive and $\lim_{t\to\infty} \phi_i(t) = 0$, we choose t_1 so that $\sum_{i=1}^n \phi_i(t) < \varepsilon < 1$ for all $t \geq t_1$. Then

$$\frac{1}{2}[D_{(1)}\ell^i(P)]_{P\in L^i} \geq \phi_i(t)\Big[\phi_i(t)\Re(f_{ii}(t)) - \sum_{j\neq i} |f_{ij}(t)|$$
$$- |g_i(t, x)| - \dot{\phi}_i(t)\Big] \geq \phi_i(t)[\phi_i(t)\Re(f_{ii}(t)) - \dot{\phi}_i(t) - \gamma(t)],$$

where $\gamma(t) = h_i(t) + \sum_{j\neq i} |f_{ij}(t)|$.

In order to have $[D_{(1)}\ell^i(P)]_{P\in L^i} > 0$ $(i = 1, \cdots, n)$ it is sufficient to choose $\phi_i(t)$ such that

$$-\dot{\phi}_i(t) + \Re(f_{ii}(t))\phi_i(t) - \gamma(t) > 0.$$

The problem is then to look for a solution $z(t)$ of $\dot{z} < \sigma(t)z - \gamma(t)$ satisfying the conditions $z(t) > 0$ $(t \geq t_1)$, $\lim_{t\to\infty} z(t) = 0$, knowing that

$$\int_t^\infty \gamma(v)\left[\exp \int_v^t \sigma(s)\, ds\right] dv \to 0 \text{ as } t \to \infty .$$

If $w(t)$ satisfies $\dot{w}(t) = \sigma(t)w(t) - \gamma(t)$, it follows that $z(t) = 2w(t)$ satisfies the differential inequality $\dot{z}(t) < \sigma(t)z(t) - \gamma(t)$. It is then sufficient to find a solution $w(t)$ for which $w(t) > 0$ $(t \geq t_1)$ and $\lim_{t\to\infty} w(t) = 0$. The solution

$$w(t) = \exp\left(\int_{t_1}^{t} \sigma(s)\, ds\right) \cdot \int_t^\infty \gamma(v)\left[\exp\left(-\int_{t_1}^{v} \sigma(s)\, ds\right)\right] dv = \int_t^\infty \gamma(v)\left[\exp \int_v^t \sigma(s)\, ds\right] dv$$

exists and $w(t) \to 0$ as $t \to \infty$.

Since $[D_{(1)}\, m^1(P)]_{P\in M^1} = -1$, it follows from Ważewski's Second Theorem that ω is a regular polyfacial set and $S = S^* = \left[\bigcup_{i=1}^n L^i\right] - M^1$.

If we choose

$$Z = \{(t, x) \mid t = \tau > t_1,\ |x_j - y_j(\tau)| \leq \phi_j(\tau),\ j = 1, \cdots, n\},$$

it follows that

$$S \cap Z = \bigcup_{i=1}^{n} L^i \cap Z - M^1,$$

$$L^i \cap Z = \{(t, x) \mid t = \tau,\ |x_i - y_i(\tau)| = \phi_i(\tau),\ |x_j - y_j(\tau)| \leq \phi_j(\tau),\ j = 1, \cdots, n\}.$$

Therefore $Z = \prod_{j=1}^n B_j^2$, where B_j^2 is a disc in R^2, and

$$Z \cap S = \bigcup_{j=1}^{n} B_1^2 \times \cdots \times B_{j-1}^2 \times S_j^1 \times B_{j+1}^2 \times \cdots \times B_n^2,$$

where S_j^1 is the boundary of B_j^2 in R^2. Also, modulo homeomorphisms, $Z = B^{2n}$ (a solid sphere in R^{2n}) and $Z \cap S = S^{2n-1}$, the boundary of B^{2n} in R^{2n}. So $Z \cap S$ is not a retract of Z. However the function

$$\Phi: S \to S \cap Z$$

given by $\Phi(P) = P^*$, with

$$t^* = \tau, \quad x_i^* = y_i(\tau) + [x_i - y_i(\tau)]\frac{\phi_i(\tau)}{\phi_i(t)},$$

is a retraction.

Using Ważewski's First Theorem, we can conclude the existence of at least one point $P_0 = (\tau, x_0) \in Z - S$ such that

$$(t, x(t, P_0)) = I(t, P_0) \subset \omega \quad \text{for all } t \geq \tau.$$

It must be that $\beta(P_0) = \infty$ because otherwise

$$\{I(t, P_0) \mid \tau \leq t < \beta(P_0)\} \cap [\Omega - \omega] \neq \emptyset,$$

which is not possible.

Consequently, $x(t, P_0)$ is defined in the future, and $\lim_{t\to\infty} [x(t, P_0) - y(t)] = 0$. The proof of the theorem is complete.

In the sequel $U(t)$ will denote a fundamental matrix of (2) and for an $n \times n$ matrix $Z = (z_1, \cdots, z_n)$, $\|Z\|$ is defined by $\Sigma_{j=1}^{n} \|z_j\|$.

3. COROLLARIES

COROLLARY 1. *Suppose the following conditions hold:*

(i) *All solutions of* (2) *are bounded in the future.*

(ii) *In the system* (1), $g(t, x)$ *is defined on* $[T, \infty) \times \Gamma$, *where* T *is a real number and* $\Gamma = \{x \mid \|x\| < \infty\}$.

(iii) *For every constant* $M > 0$ *and some* $t_0 > T$, *there exists a continuous real-valued function* $h_M(t)$ *such that* $\int^{\infty} h_M(t) \|U^{-1}(t)\| \, dt < \infty$ *and* $\|g(t, x)\| \leq h_M(t)$ *for all* (t, x) *with* $t > t_0$, $\|x\| \leq M$.

Then for every solution $y(t)$ *of* (2) $\{$*bounded solution* $x(t)$ *of* (1)$\}$, *there exists a solution* $x(t)$ *of* (1) *defined in the future* $\{$*solution* $y(t)$ *of* (2)$\}$ *such that* $x(t) - y(t) \to 0$ *as* $t \to \infty$.

Proof. If we make the transformations $x(t) = U(t)z(t)$, $y(t) = U(t)v(t)$ in the systems (1) and (2), then

$$\dot{z}(t) = U^{-1}(t)g(t, U(t)z(t)) = f(t, z), \tag{6}$$

$$\dot{v}(t) = 0. \tag{7}$$

To prove that for every solution $y(t)$ of (2), there exists a solution $x(t)$ of (1) defined in the future such that $x(t) - y(t) \to 0$ as $t \to \infty$, it is enough to prove that for every constant $n \times 1$ matrix c there is a solution $z(t)$ of (6) defined in the future with $z(t) \to c$ as $t \to \infty$.

There exists a constant K such that $\|U(t)\| \leq K$ for $t \geq t_0$. Therefore

$$\|f(t, z)\| \leq \|U^{-1}(t)\| \cdot \|g(t, U(t)z)\| \leq \|U^{-1}(t)\| h_{\tilde{M}}(t) \quad (t \geq t_0),$$

for $\|z\| \leq M$, where $\tilde{M} = KM$; and by hypothesis, $\int^{\infty} \|U^{-1}(t)\| h_{\tilde{M}}(t) \, dt < \infty$.

Now the existence of a solution $z(t)$ of (6) with the required property follows from Theorem 1 applied to the systems (6) and (7).

If $x(t)$ is a given bounded solution of (1), we consider the solution $\tilde{y}(t)$ of (2) defined by the integral equation

$$x(t) = \tilde{y}(t) + \int_{t_1}^{t} U(t)U^{-1}(s)g(s, x(s)) \, ds \quad (t_1 \geq t_0),$$

such that $\|x(t)\|$ is less than some constant M for all $t \geq t_1$.

Clearly,

$$\|U^{-1}(s)g(s, x(s))\| \leq \|U^{-1}(s)\| h_M(s) \qquad (s \geq t_1).$$

Therefore,

$$\int^{\infty} \|U^{-1}(s)g(s, x(s))\| \, ds < \infty, \quad \text{and}$$

$$x(t) = \tilde{y}(t) + U(t)\int_{t_1}^{\infty} U^{-1}(s)g(s, x(s))\,ds + U(t)\int_{\infty}^{t} U^{-1}(s)g(s, x(s))\,ds$$

$$= y(t) + U(t)\int_{\infty}^{t} U^{-1}(s)g(s, x(s))\,ds,$$

where $y(t)$ is a solution of (2).

It follows that $x(t) - y(t) \to 0$ as $t \to \infty$. The proof of the Corollary 1 is complete.

We notice that under the hypotheses of Corollary 1 there are systems (1) for which one can find unbounded solutions. For instance $x = \exp t$ is a solution of $\dot{x} = [\exp(-t)]x^2$.

COROLLARY 1'. *Suppose assumptions* (i) *and* (iii) *of Corollary* 1 *hold and further, suppose that:*

(iii') *For every constant* $M > 0$ *and some* $t_0 > T$ *there exists a continuous real-valued function* $h_M(t)$ *such that*

$$\int^{\infty} h_M(t)\left[\exp \int_{t}^{t_0} \sum_{j=1}^{n} \Re(f_{jj}(s))\,ds\right] dt \; dt < \infty$$

and $\|g(t, x)\| \leq h_M(t)$ *for all* (t, x) *with* $t \geq t_0$, $\|x\| \leq M$.

Then the conclusions of Corollary 1 *hold.*

Proof. From the Jacobi-Liouville formula,

$$\det U(t) = \det U(t_0)\left[\exp \int_{t_0}^{t} \sum_{j=1}^{n} f_{jj}(s)\,ds\right],$$

it follows that

$$|[\det U(t)]^{-1}| = |[\det U(t_0)]^{-1}| \left[\exp \int_{t}^{t_0} \sum_{j=1}^{n} \Re(f_{jj}(s))\,ds\right].$$

Since $U^{-1}(t) = [\det U(t)]^{-1} \operatorname{adj} U(t)$ and hypothesis (i) implies $\operatorname{adj} U(t)$ is bounded, it is clear that

$$\|U^{-1}(t)\| = |[\det U(t)]^{-1}|\cdot\|\operatorname{adj} U(t)\|$$

$$= |[\det U(t_0)]^{-1}|\cdot\|\operatorname{adj} U(t)\|\cdot\left[\exp\int_t^{t_0}\sum_{j=1}^{n}\Re(f_{jj}(s))\,ds\right]$$

$$\leq K\left[\exp\int_t^{t_0}\sum_{j=1}^{n}\Re(f_{jj}(s))\,ds\right],$$

for some constant K. Therefore, $\int^{\infty} h_M(t)\|U^{-1}(t)\|dt < \infty$ and Corollary 1' now follows immediately from Corollary 1.

In the system (1) suppose now

$$g_i(t, x) = \sum_{j=1}^{m_i} g_{ij}(t, x),\quad |g_{ij}(t, x)| \leq F_i(t)\|x\|^{\alpha_{ij}},\ \alpha_{ij} \geq 0,\ \alpha = \max(\alpha_{ij}).$$

COROLLARY 2. *Suppose* $\Gamma = \{x \mid \|x\| < \infty\}$, $y = y(t)$ *is a solution of* (2), *and*

$$\int_t^{\infty} F_j(v)\left[\exp\int_v^t \Re(f_{jj}(s))\,ds\right]dv \to 0 \quad \textit{as } t\to\infty \tag{8}$$

$$(j = 1, \cdots, n),$$

$$\int_t^{\infty} \|y(t)\|^{\alpha} F_j(v)\left[\exp\int_v^t \Re(f_{jj}(s))\,ds\right]dv \to 0 \quad \textit{as } t\to\infty \tag{9}$$

$$(j = 1, \cdots, n),$$

$$\int_t^{\infty} |f_{ij}(v)|\left[\exp\int_v^t \Re(f_{ii}(s))\,ds\right]dv \to 0 \quad \textit{as } t\to\infty \quad (i \neq j). \tag{10}$$

Then there exists a solution $x(t)$ of (1) such that $x(t) - y(t) \to 0$ as $t\to\infty$.

Proof. If we choose a $t_0 > T$ and $\varepsilon = 1$, then the statement $(t, x) \in W_{\varepsilon,t_0,y}$ implies that $\|x\| < \|y(t)\| + 1$, and that

$$|g_i(t, x)| \leq \sum_{j=1}^{m_i}|g_{ij}(t, x)| \leq \sum_{j=1}^{m_i} F_i(t)\|x\|^{\alpha_{ij}} \leq \sum_{j=1}^{m_i} F_i(t)[1 + \|y(t)\|]^{\alpha_{ij}}$$

$$\leq K_1 F_i(t)[1 + \|y(t)\|]^{\alpha} \leq K_1 F_i(t)[2\sup(1, \|y(t)\|)]^{\alpha}$$

$$\leq K_2 F_i(t)[1 + \|y(t)\|^{\alpha}],$$

where K_1 and K_2 are constants.

Now we use Theorem 1 with $h_i(t) = K_2 F_i(t)[1 + \|y(t)\|^{\alpha}]$ and obtain the desired conclusions.

COROLLARY 2'. *Suppose* $\Gamma = \{x \mid \|x\| < \infty\}$,

(11) $$K \leq \int_t^v \Re(f_{ii}(s))\,ds, \quad \textit{for some constant } K \textit{ and all } v \geq t \geq T \quad (i = 1, \cdots, n),$$

(12) $$\int^\infty |f_{ij}(v)| \left[\exp \int_T^v \Re(f_{kk}(s))\,ds\right] dv < \infty \quad (i \neq j,\ k = 1, \cdots, n),$$

(13) $$\int^\infty F_j(v) \left[\exp \int_T^v \alpha \Re(f_{ii}(s))\,ds\right] dv < \infty \quad (i, j = 1, \cdots, n).$$

Then for every solution $y(t)$ *of* (2) *there exists a solution* $x(t)$ *of* (1) *such that* $(x(t) - y(t) \to 0$ *as* $t \to \infty$.

Proof. Consider the system

(14) $$\dot{z}_i = f_{ii}(t) z_i \qquad (i = 1, \cdots, n)$$

and rewrite (2) in the form

(2) $$\dot{y}_i = f_{ii}(t) y_i + \sum_{j \neq i} f_{ij}(t) y_j .$$

Hypotheses (11), (12) and (13) imply that

$$\int^\infty |f_{ij}(v)|\,dv < \infty \ (i \neq j) \ \text{and} \ \int^\infty F_k(v)\,dv < \infty, \quad (k = 1, \cdots, n).$$

By applying the Corollary 2 to the systems (2) and (14) in relation to any solution $z(t)$ of (14), we conclude that there exists a solution $y(t)$ of (2) such that $z(t) - y(t) \to 0$. Hence, for the fundamental matrix $z(t)$ of (14) defined by the conditions

$$(Z(t))_j^i = 0 \text{ if } i \neq j \text{ and } (Z(t))_i^i = \exp \int_T^t f_{ii}(s)\,ds,$$

there exists a matrix $Y(t)$ of solutions of (2) such that $Y(t) - Z(t) \to 0$ as $t \to \infty$. As there exists an $\varepsilon > 0$ such that $\exp \int_T^t \Re(f_{ii}(s))\,ds > \varepsilon$ $(i = 1, \cdots, n;\ t \geq T)$, the existence of a $\delta > 0$ and a $t_0 \geq T$ such that $|(Y(t))_i^i| > \delta$ for $t \geq t_0$ $(i = 1, \cdots, n)$ and $(Y(t))_j^i \to 0$ as $t \to \infty$ for $i \neq j$ follows. This implies that $Y(t)$ is a fundamental matrix of (2). Therefore, for every solution $y(t)$ of (2) there exists a solution $z(t)$ of (1) such that $z(t) - y(t) \to 0$ as $t \to \infty$.

So, if t_1 is sufficiently large and if $t \geq t_1$, then

$$\|y(t)\| \leq \sum_{j=1}^{n} K_j \exp \int_T^t \Re(f_{jj}(s))\, ds + 1$$

$$\leq [n+1] \sup_{1 \leq j \leq n} \left\{ K_j \exp \int_T^t \Re(f_{jj}(s))\, ds,\ 1 \right\}$$

where the $K_j \geq 0$ are constants. Thus there exists a constant $K \geq 0$ such that

$$\|y(t)\|^\alpha \leq K \left[1 + \sum_{j=1}^{n} \exp \int_T^t \alpha \Re'f_{jj}(s))\, ds \right],$$

$$\int_t^\infty F_k(v) \left[\exp \int_v^t \Re(f_{kk}(s))\, ds \right] dv \leq \text{constant} \int_t^\infty F_k(v)\, dv \to 0 \quad (t \to \infty),$$

$$\int_t^\infty \|y(v)\|^\alpha F_k(v) \left[\exp \int_v^t \Re(f_{kk}(s))\, ds \right] dv \leq \int_t^\infty \|y(s)\|^\alpha F_k(s)\, ds$$

$$\leq \text{constant} \left[\int_t^\infty F_k(v)\, dv + \sum_{j=1}^{n} \int_t^\infty F_k(v) \left\{ \exp \int_T^v \alpha \Re(f_{jj}(s))\, ds \right\} dv \right] \to 0$$

$$(t \to \infty),$$

$$\int_t^\infty |f_{ij}(v)| \left[\exp \int_v^t \Re(f_{ii}(s))\, ds \right] dv \leq \int_t^\infty |f_{ij}(s)|\, ds \to 0 \quad (t \to \infty),$$

where $i \neq j$ $(k = 1, \cdots, n)$.

Corollary 2' follows now from Corollary 2.

We notice that Corollary 2' is a generalization of Theorem III-2 in [2, p. 1524]. Also it is easy to obtain Theorem III-3, [2, p. 1526] as a consequence of Corollary 2'.

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