ON A CERTAIN CLASS OF TRANSFORMATION GROUPS

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1. INTRODUCTION

This note is intended as an appendix to the author's Chapter XV of [1]. The main result [1, Chapter XV, 1.4] of the latter will be referred to in the present note as the CDT (complementary dimension theorem), and it is, in fact, precisely the case $m = 1$ of the main result of this note.

The prototype of the class of transformation groups that we shall study can be described as follows. For each $i = 1, 2, \cdots, m$, let G_i be a closed subgroup of $SO(k_i + 1)$ such that G_i is transitive on the sphere S^{k_i} in the usual action. Let M_i be euclidean space of dimension $k_i + 1$ for $i = 1, 2, \cdots, m$, and let M_0 be euclidean space of some arbitrary dimension. Let $M = M_0 \times M_1 \times \cdots \times M_m$, let n be the dimension of M, and let $G = G_1 \times G_2 \times \cdots \times G_m$. Define an action of G on M as follows. If $g = (g_1, g_2, \cdots, g_m)$, where $g_i \in G_i$, and if $x = (x_0, x_1, \cdots, x_m)$, where $x_i \in M_i$, then let $g(x) = (x_0, g_1(x_1), g_2(x_2), \cdots, g_m(x_m))$. In this note we shall show that, at least as far as cohomology is concerned, the transformation group (G, M) described above is essentially characterized by the fact that

$$\dim F(G_i, M) = n - k_i - 1 .$$

(For the precise statement, see Theorems 1.1 and 4.1 below.)

We shall use the notation of [1]; dim and $\dim_p$ will denote cohomology dimension over Z and over Z_p, respectively (see [1, Chapter I, Section 1.2]). The notation n-cm will be used for $n\text{-cm}_Z$ (see [1, I, Section 3]). If X is an n-cm with boundary B, we shall say that a transformation group on X satisfies the hypotheses of the CDT or of Theorem 1.1 if it does so for the naturally related action on X^{dB} (see [1, XV, Section 1.2]). If G acts on a space X and if $Y \subset X$, then we denote by Y* the image of Y in the orbit space $X^* = X/G$. If K is a subgroup of G, then we denote the identity component of K by K^0, and the normalizer of K in G by $N_G(K)$ (or by N(K) if no confusion can arise).

If G is a compact Lie group acting on a space M, and K is a subgroup of G, we let $M_K = \{y \in M \mid G_y \sim K\}$, which is the set of points with orbits of type (K) (see [1, VIII, Section 2]). If M is an n-cm, we denote a principal isotropy group by H. If furthermore $G = G_1 \times G_2 \times \cdots \times G_m$ and $I \subset \{1, 2, \cdots, m\}$, we let

$$H_I = \prod_{i \in I} (G_i \cap H) \times \prod_{i \notin I} G_i \qquad \text{and} \qquad M_I = \bigcup_{J \subset I} M_{H_J}.$$

We also denote by m(I) the number of elements of $\{1, 2, \cdots, m\} - I$.

If $G_1 \times G_2 \times \cdots \times G_m$ is a compact Lie group acting on the n-cm M, we shall say that condition (A) is satisfied if each of the following three statements is true:

(i) Each G_i is effective on M.

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(ii) There are no indices i and j such that $G_i \approx G_j \approx Z_2$ and such that the diagonal subgroup acts trivially on M.

(iii) There are no indices i and j such that either $G_i^0 \approx G_j^0 \approx SO(2)$ or $G_i^0 \approx G_j^0 \approx Sp(1)$ and such that the action of $G_i^0 \times G_j^0$ on M has the diagonal subgroup as a principal isotropy subgroup.

Remark. The situation that we actually wish to rule out by (ii) and (iii) is the case in which

$$G_i/(G_i \cap H) \approx (G_i \times G_j)/((G_i \times G_j) \cap H) \approx G_j/(G_j \cap H)$$

and is an integral cohomology sphere. Knowledge of the homogeneous integral cohomology spheres (see [2] and [4]) shows easily that this can happen only if condition (A) is violated. We shall assume this fact without proof. If the reader wishes, he may replace condition (A) by this seemingly stronger restriction. See also Remark (a) below Theorem 1.1.

We can now state the main theorem of this note.

THEOREM 1.1. *Let* $G = G_1 \times G_2 \times \cdots \times G_m$ *act on the separable* n-cm M *of finite covering dimension, and assume that condition* (A) *is satisfied. Let* $k_i = \max\{\dim G_i(y) \mid y \in M\}$, *and put* $k = \Sigma k_i$. *Assume that*

$$\dim(F(G_i, M), x) = n - k_i - 1$$

for some $x \in F(G, M)$ *and all* $i = 1, 2, \cdots, m$. *Then, locally at* x, *the following statements hold:*

(1) *The orbit types of* G *are exactly the* (H_I).

(2) $H^*(G_i/(G_i \cap H); Z) \approx H^*(S^{k_i}; Z)$, *so that* $H^*(G/H_I; Z) \approx H^*(\prod_{i \in I} S^{k_i}; Z)$.

(3) M_I^* *is an* $(n - k - m(I))$-cm *with boundary* $\bigcup_{J<I} M_J^*$ (*where* $<$ *denotes proper inclusion*).

(4) *There exists a cross-section* $f: M/G \to M$ *such that* $f(M_{H_I}^*) \subset F(H_I, M)$.

Remarks. (a) Condition (A) is not an essential restriction, since if such a pair (i, j) did occur one could group $G_i \times G_j$ together into one factor, and (even after making this new factor effective) the new decomposition of G into a direct product would still satisfy the other hypotheses of the theorem.

(b) Note the particular cases $I = \emptyset$ and $I = \{1, 2, \cdots, m\}$ of conclusion (3). The case $I = \emptyset$ says that F(G, M) is an $(n - k - m)$-cm, and the case $I = \{1, 2, \cdots, m\}$ says that M/G is an $(n - k)$-cm with boundary consisting of the singular orbits.

(c) Conclusion (1) implies that G is effective, since each G_i is assumed to be effective.

(d) Conclusion (2) follows immediately from the CDT, and we include it here merely for completeness. Also, note that in fact $G_i/(G_i \cap H) \approx S^{k_i}$, unless it is of the form SO(3) modulo the icosahedral subgroup (see [2]).

(e) The condition on the cross-section is precisely what we need if we wish to reconstruct M and the action of G from M/G and the knowledge of the isotropy groups of the orbits.

(f) Note that we do not require G to be connected, so that, in particular, the theorem is applicable to the case $G_i \approx Z_2$ $(i = 1, 2, \cdots, m)$ and $\dim F(G_i, M) = n - 1$. In fact, Lemma 2.2 is essentially Theorem 1.1 for $G = Z_2 \times Z_2$.

2. PRELIMINARY RESULTS

LEMMA 2.1. *Suppose that* G *is a compact Lie group that satisfies the hypotheses of the* CDT *at* x *on the* n-cm M. *Let* $A \subset M$ *be an* m-cm *which is invariant under* G *and such that* $x \in A$ *and* $A \not\subset F = F(G, M)$. *Then* G *satisfies the hypotheses of the* CDT *at* x *on* A.

Proof. From the CDT it follows that if H is the principal isotropy group of G on M, then

$$A - F \approx \frac{A - F}{G} \times \frac{G}{H},$$

and hence $\frac{A - F}{G}$ is an $(m - k)$-cm, where $k = \dim G/H$. Thus

$$H_c^{m-k}(A - F; Z_2) \neq_{Lx} 0$$

(see [1, XV, Definition 2.2] for the notation), and by the exact cohomology sequence of A mod F, we see that

$$\dim(F \cap A, x) \geq \dim_2(F \cap A, x) \geq m - k - 1,$$

as was to be shown.

LEMMA 2.2. *Let* X *be a space, and let* $A \subset X$ *be a closed set such that* X^{dA} *is an* n-cm *with boundary* B'. *Let* $B = X \cap B'$. *Then* X *is an* n-cm *with boundary* $A \cup B$, *and both* A *and* B *are* $(n - 1)$-cms *with boundary* $A \cap B$.

Proof. Since the boundary points of a cm with boundary are topologically distinguishable from the nonboundary points (for example, by the local cohomology groups), we see easily that $B' = B^{d(A \cap B)}$, and hence B is an $(n - 1)$-cm with boundary $A \cap B$.

Let $A' = A^{d(A \cap B)}$; then, since $(X^{dB})^{dA'} \approx (X^{dA})^{dB'}$ is an n-cm, we see that A' is an $(n - 1)$-cm, and hence A is an $(n - 1)$-cm with boundary $A \cap B$.

It also follows from the relation $(X^{dB})^{dA'} \approx (X^{dA})^{dB'}$ that X^{dB} is an n-cm with boundary $A' = A^{d(A \cap B)}$. Also, since A and B are $(n - 1)$-cms with common boundary $A \cap B$, it follows that $A \cup B$ is an $(n - 1)$-cm (see [6]; the proof of Lemma 2.3 in [1, XV] is also easily seen to yield this result). Thus, by [1, XV, Lemma 2.3], to show that X is an n-cm with boundary $A \cup B$ we need only show that $H_c^*(X) =_{Lx} 0$ for $x \in A \cup B$. Since $X - (A \cap B)$ is an n-cm with boundary $(A \cup B) - (A \cap B)$, it is sufficient to do this for $x \in A \cap B$. But if we let $X^{dA} = X \cup X'$, and $X \cap X' = A$, then in the Mayer-Vietoris sequence

$$\cdots \to H_c^i(X \cup X') \to H_c^i(X) \oplus H_c^i(X') \to H_c^i(A) \to \cdots$$

we have $H_c^i(X \cup X') =_{Lx} 0$ and $H_c^i(A) =_{Lx} 0$ since x is in the boundary of both of the cms X^{dA} and A. Thus $H_c^i(X) =_{Lx} 0$, as was to be shown.

LEMMA 2.3. *Let* X_1 *and* X_2 *be* n-cms *with boundaries* B_1 *and* B_2, *respectively, with* $X_1 \subset X_2$ *and* $B_1 \subset B_2$. *Then* X_1 *is an open subset of* X_2.

Proof. Since X_1 and X_2 are locally compact, it is easy to see that we may assume X_1 to be closed in X_2. Let $x \in X_1 - B_1$, and let U be a connected open neighborhood of x in X_2, with $U \cap B_1 = \emptyset$. Then $U \cap X_1$ is an n-cm that is a closed subset of the connected n-cm U (respectively, $U^{d(U \cap B_2)}$ if $x \in B_2$). Since $H_c^n(U \cap X_1) \neq 0$, it follows that $U \cap X_1$ cannot be a proper subset of U [1, I, Theorem 4.3] (respectively of $U^{d(U \cap B_2)}$, whence $x \notin B_2$). Thus, $B_2 \cap X_1 = B_1$, and $X_1 - B_1$ is open in X_2. Since $B_2 \cap X_1 = B_1$, we may regard $X_1^{dB_1}$ as a subset of $X_2^{dB_2}$, and as above we see that $X_1^{dB_1}$ is open in $X_2^{dB_2}$. It follows that X_1 is open in X_2, as claimed.

3. PROOF OF THE MAIN THEOREM

All statements in this section are to be interpreted as holding locally at x, where x is a given point at which the hypotheses of Theorem 1.1 are satisfied.

We shall prove Theorem 1.1 by induction on m. For m = 1 it is precisely the CDT, and hence is true. For the time being, we shall restrict our attention to the case m = 2.

Then, by Lemma 2.1 applied to $A = F(G_2, M)$, we must have one of the following possibilities:

Case (a): $F(G_1, M) \supset F(G_2, M)$.

Case (b): $\dim F(G_1, F(G_2, M)) = (n - k_2 - 1) - k_1 - 1 = n - (k_1 + k_2) - 2$.

First note that if the condition of case (b) holds, then

$$\dim F(G_1, F(G_2, M)) < \dim F(G_1, M),$$

and hence $F(G_1, M) \not\subset F(G_2, M)$. Thus, if the condition of case (b) holds, then it must also hold with G_1 and G_2 in the opposite order. Therefore, in case (a),

$$F(G_1, M) = F(G_2, M).$$

Assuming for the moment that the condition of case (a) holds, we see that the action of G_1 on M/G_2 leaves the boundary $F(G_2, M)$ of the $(n - k_2)$-cm M/G_2 pointwise stationary. Thus, in fact, G_1 must leave all of M/G_2 stationary (see [1, XV, Lemma 2.1]). It follows that $M/G_1 \approx M/G \approx M/G_2$ (naturally), and hence that

$$G_1/(G_1 \cap H) \approx G/H \approx G_2/(G_2 \cap H) \text{ (naturally).}$$

But this is an integral cohomology sphere, by the CDT, contrary to the assumption that condition (A) is satisfied. Thus case (a) does not arise.

We now consider case (b). Here

$$\dim \frac{F(G_1, M)}{G_2} = (n - k_1 - 1) - k_2 = n - (k_1 + k_2) - 1,$$

by Lemma 2.1 and the CDT. Also, $F(G_1, M)/G_2 \subset F(G_1, M/G_2)$. Thus we see that

$$\dim F(G_1, F(G_2, M)) = n - (k_1 + k_2) - 2 < n - (k_1 + k_2) - 1$$

$$= \dim (F(G_1, M)/G_2) \leq \dim F(G_1, M/G_2) .$$

But this implies that the action of G_1 on M/G_2 satisfies the hypotheses of the CDT. Thus

$$F(G_1, (M/G_2)^{dF(G_2,M)}) \approx F(G_1, M/G_2)^{dF(G_1,F(G_2,M))}$$

is an $(n - (k_1 + k_2) - 1)$-cm, and hence $F(G_1, M/G_2)$ is an $(n - (k_1 + k_2) - 1)$-cm with boundary $F(G_1 \times G_2, M)$. Also, G_2 satisfies the hypotheses of the CDT on $F(G_1, M)$, so that $F(G_1, M)/G_2$ is an $(n - (k_1 + k_2) - 1)$-cm with boundary $F(G_1 \times G_2, M)$. Thus, since

$$F(G_1, M)/G_2 \subset F(G_1, M/G_2) ,$$

it follows from Lemma 2.3 that these sets are identical (near x).

Moreover, comparing the actions of G_2 on $F(G_1, M)$ and on M, and using the CDT, we see that G_2 is effective on $F(G_1, M)$, since it is effective on M.

LEMMA 3.1. *With the hypotheses of Theorem* 1.1 (*respectively, and if* $k_i = k_j > 0$), *it cannot happen that* G_i *and* G_j (*respectively,* G_i^0 and G_j^0) *are isomorphic and that* $G_i \times G_j$ (*respectively,* $G_i^0 \times G_j^0$) *possesses the diagonal subgroup as an isotropy group.*

Proof. In the first case, any point with such an isotropy group would clearly represent a point of $F(G_i, M/G_j)$ that is not in $F(G_i, M)/G_j$, contrary to the equality (proved above) of these sets. If $k_i > 0$ and $k_j > 0$, then the group $G_i^0 \times G_j^0$ still satisfies the hypotheses of Theorem 1.1. Thus the case in parentheses follows from the first case.

We now proceed with the proof of Theorem 1.1 in full generality. If I is a subset of $\{1, 2, \cdots, m\}$, we put $I' = \{1, 2, \cdots, m\} - I$,

$$G_I = \prod_{i \in I} G_i , \qquad k(I) = \sum_{i \in I} k_i .$$

LEMMA 3.2. *If* $I \neq \emptyset$, *then the action of* G_I *on* $F(G_{I'}, M)$ *and the action of* G_I *on* $M/G_{I'}$ *both satisfy the hypotheses of Theorem* 1.1.

Proof. The lemma makes sense, because of the inductive assumption. If $I' = \emptyset$, the lemma is trivial. By an easy induction, it suffices to treat the case $I = \{1, 2, \cdots, m - 1\}$. Then the treatment of the case $m = 2$ of Theorem 1.1 shows that each G_i is effective on $F(G_m, M)$ for $i \in I$. It also shows that the hypotheses of Theorem 1.1 are satisfied, with the possible exception of condition (A). But if $G_i \approx G_j$ (respectively, $k_i = k_j > 0$ and $G_i^0 \approx G_j^0$) and $G_i \times G_j$ (respectively, $G_i^0 \times G_j^0$) has the diagonal subgroup as a principal isotropy group on $(M/G_m)^{dF(G_m,M)}$, then it also does on $F(G_m, M)$ (for otherwise G_i could not be effective on $F(G_m, M)$). But then $G_i \times G_j$ (respectively, $G_i^0 \times G_j^0$) has the diagonal subgroup as an isotropy group (not necessarily principal) on M, contrary to Lemma 3.1.

COROLLARY 3.3. $F(G_I, M/G_{I'}) = F(G_I, M)/G_{I'}$.

Proof. Clearly $F(G_I, M/G_{I'}) \supset F(G_I, M)/G_{I'}$. Also, the corollary is trivial if I or I' is empty. Thus suppose $I \neq \emptyset$. Then, by the inductive assumption, $M/G_{I'}$ is an $(n - k(I'))$-cm with boundary

$$B = \bigcup_{J < I'} F(G_J, M)/G_{I'}.$$

Hence, by Lemma 3.2 and the inductive assumption, we see that $F(G_I, M/G_{I'})$ is an $(n - k - m(I))$-cm with boundary $F(G_I, B)$.

But also $F(G_I, M)/G_{I'}$ is an $(n - k - m(I))$-cm with boundary

$$\bigcup_{J < I'} F(G_J, F(G_I, M))/G_{I'} \subset F(G_I, B).$$

Thus the corollary follows from Lemma 2.3.

Let $\pi_i: G \to G_i$ be the natural projection. Conclusion (1) of Theorem 1.1 follows from the CDT and the following lemma.

LEMMA 3.4. *If* $K = G_y$ *for some point* y *near* x, *then* $K \cap G_i = \pi_i(K)$, *so that* $K = \Pi(K \cap G_i)$.

Proof. We may assume that $H \subset K$. First suppose that $y \in F(G_j, M)$ for some j, and let $G_j^! = \prod_{i \neq j} G_i$. Then, by Lemma 3.2 applied to the action of $G_j^!$ on $F(G_j, M)$ and by the inductive assumption, we see that Lemma 3.4 holds for $(G_j^!)_y$. But $G_y = (G_j^!)_y \times G_j$, and the result follows.

On the other hand, suppose that $y \notin F(G_j, M)$ for any j. Then by Corollary 3.3, $G_i^!(y) \notin F(G_i, M/G_i^!)$. But the action of G_i satisfies the hypotheses of the CDT on M, $F(G_i^!, M)$, and on $M/G_i^!$, by Lemma 3.2. Thus $G_i \cap K = G_i \cap H$, since this is a principal isotropy group for the action of G_i on M. Also, the isotropy group of G_i at the point $G_i^!(y) \in M/G_i^!$ is $\pi_i(K)$. But, by the CDT, G_i has exactly two types of orbits on $M/G_i^! \supset F(G_i^!, M)$, and it follows that this isotropy group must be principal and that it occurs as the principal isotropy group for G_i on $F(G_i^!, M)$ and hence on M. Thus $\pi_i(K) = G_i \cap H = G_i \cap K$ (since $K \supset H$). This completes the proof of the lemma.

We shall now prove conclusion (3) of 1.1. Note that

$$M_I^* \approx \frac{F(G_{I'}, M)}{G_I} \text{ (naturally)}$$

For $\emptyset \neq I \neq \{1, \cdots, m\}$, the conclusion follows easily by the inductive assumption. If $I = \emptyset$, then

$$M_I^* = F(G, M) = F(G_1, F(G_2 \times \cdots \times G_m, M)),$$

and the conclusion follows from Lemma 3.2 and the inductive assumption. If $I = \{1, \cdots, m\}$, then

$$M_I^* = \frac{M}{G} \approx \frac{M/G_1}{G_2 \times \cdots \times G_m} \text{ (naturally)},$$

and by Lemma 3.2 and the inductive assumption we see that

$$\frac{(M/G_1)^{dF(G_1,M)}}{G_2 \times \cdots \times G_m} \approx \left(\frac{M}{G}\right)^{dF(G_1,M)^*}$$

is an (n - k)-cm with boundary

$$B' = \bigcup_{i=2}^{m} F(G_i, (M/G_1)^{dF(G_1,M)})^* \approx \left[\bigcup_{2}^{m} F(G_i, M/G_1)^*\right]^{dF(G_1,M)^*}$$

$$= \left[\bigcup_{2}^{m} F(G_i, M)^*\right]^{dF(G_1,M)^*}$$

(the last equality holds by Corollary 3.3 applied to $G_i \times G_1$). Thus by Lemma 2.2, M/G is an (n - k)-cm with boundary

$$F(G_1, M)^* \cup \bigcup_{2}^{m} F(G_i, M)^* = \bigcup_{1}^{m} F(G_i, M)^* = \bigcup M_J^*$$

(J running over the proper subsets of $\{1, \cdots, m\}$), as was to be shown.

It remains to prove conclusion (4) of 1.1. Let $G = G_J \times G_{J'}$ be any splitting of G chosen once and for all. Then, by the inductive assumption, there exist cross-sections (locally at x)

$$\frac{M}{G} \approx \frac{M/G_J}{G_{J'}} \overset{f_2}{\rightarrow} \frac{M}{G_J} \overset{f_1}{\rightarrow} M$$

that satisfy the requirements of (4) for the actions of $G_{J'}$ on M/G_J and of G_J on M, respectively.

Let y be in the image of $f = f_1 \circ f_2$. We know, by (1), that there is a $g_0 \in G$ and an $I \subset \{1, \cdots, m\}$ such that

$$H_I = g_0^{-1} G_y g_0 = G_{g_0^{-1}(y)} .$$

Let $K = H_I$ and $z = g_0^{-1}(y)$. We shall identify $G(y) = G(z)$ with G/K by the natural map $gK \rightarrow g(z)$. Thus y corresponds to the coset g_0K. We must show that $y \in F(K, M)$, and hence that

$$g_0 K \in F(K, G/K) = N(K)/K .$$

We let $K_J = (K \cap G_J)$ and $K_{J'} = (K \cap G_{J'})$, so that $K = K_J \times K_{J'}$ by (1). By our identification we have the situation

$$\frac{G}{G} = \frac{G_J}{G_J} \times \frac{G_{J'}}{G_{J'}} \underset{p_2}{\overset{f_2}{\rightleftarrows}} \frac{G_J}{G_J} \times \frac{G_{J'}}{K_{J'}} \underset{p_1}{\overset{f_1}{\rightleftarrows}} \frac{G_J}{K_J} \times \frac{G_{J'}}{K_{J'}} = \frac{G}{K},$$

where $f_1 f_2(G/G)$ is the point $g_0 K \in G/K$, and where p_1 and p_2 are the natural projections. Since the map f_2 satisfies condition (4), we see that

$$p_1(g_0 K) = f_2(G/G) \in \frac{G_J}{G_J} \times \frac{N(K_{J'})}{K_{J'}},$$

and hence $g_0 K \in \frac{G_J}{K_J} \times \frac{N(K_{J'})}{K_{J'}}$. Also, since the map f_1 satisfies (4),

$$g_0 K = f_1(f_2(G/G)) \in \frac{N(K_J)}{K_J} \times \frac{G_{J'}}{K_{J'}}.$$

Thus

$$g_0 K \in \frac{N(K_J)}{K_J} \times \frac{N(K_{J'})}{K_{J'}} = \frac{N(K)}{K},$$

as was to be shown.

4. APPLICATIONS

As in [1, XV, 1.6] the local theorem above yields immediately a corresponding global theorem:

THEOREM 4.1. *Let* $G = G_1 \times G_2 \times \cdots \times G_m$ *act on the separable* n-cm M *of finite covering dimension, and assume that condition* (A) *is satisfied. Assume that* $H^*(M; Z) \approx H^*(S^n; Z)$ *and that* $\dim F(G_i, M) = n - k_i - 1$ *for all* i (*notation of Theorem* 1.1). *Then the conclusions of Theorem* 1.1 *hold globally. Moreover,* M_I^* *is acyclic for* $I \neq \emptyset$, *and* $H^*(F(G, M); Z) = H^*(S^{n-k-m}; Z)$.

This may be proved by applying Theorem 1.1 to the action of G on the open cone over M. The details, along the lines of the proof of [1, XV, Corollary 1.6], are easy and will be left to the reader.

Theorem 1.1 is applicable to a situation considered by Montgomery and Mostow in [5], and it strengthens their results somewhat (mainly by showing that the cross-section obtained in [5, Section 6] can be taken to be a cross-section for *all* of the orbits). We shall state and prove the local analogue. It is clear that the globalization technique applies to this case.

THEOREM 4.2. *Let* T *be an* m-*dimensional toral group acting effectively on a separable* n-cm M *of finite covering dimension with* $\dim(F(T, M), x) = n - 2m$. *Then there exist circle subgroups* $T_1, T_2, \cdots, T_m$ *of* T *such that* T *is the direct product of the* T_i *and* $\dim(F(T_i, M), x) = n - 2$ *for all* $i = 1, 2, \cdots, m$. *Thus the hypotheses of Theorem* 1.1 *hold for the action of* T *on* M (*near* x).

Proof. All statements below should be interpreted as holding locally at x. We shall use induction on m. Say that T' is a torus of codimension one in T. Then $n - 2(m - 1) \geq \dim F(T', M)$ by [3, Lemma 2.1], and if $\dim F(T', M) > \dim F(T, M)$, then $\dim F(T', M) \geq n - 2m + 2$ by dimensional parity [1, V, Theorem 3.2]. Thus, for such a torus T', we have $\dim F(T', M) = n - 2m + 2$. Therefore, if j is the number of subtori T' of codimension one with $\dim F(T', M) > n - 2m$, then Borel's formula [1, XIII, Theorem 4.3] reads

$$2m = n - (n - 2m) = j((n - 2m + 2) - (n - 2m)) = 2j,$$

and hence $j = m = \dim T$.

By the inductive assumption, we see that if $T' \neq T''$ are subtori of the above type, then by Theorem 1.1 we see that there are *exactly* $m - 1$ circle subgroups of T' (respectively, of T'') with fixed point sets of dimension $n - 2$, and that these subgroups span T' (respectively, T''). It follows that there are at most $m - 2$ of these circle subgroups contained in $T' \cap T''$ so that T'' must contain such a subgroup *not* contained in T'. Thus there are (at least) m circle subgroups $T_1, T_2, \cdots, T_m$ spanning T and with $\dim F(T_i, M) = n - 2$.

Let T^* be the *direct* product of the T_i $(i = 1, 2, \cdots, m)$, and let $\phi: T^* \to T$ be the natural homomorphism (with finite kernel). Then Theorem 1.1 applies to the induced action of T^* on M and, in particular, it implies that T^* is effective. Thus ϕ is an isomorphism, and the result follows.

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