NIL IDEALS IN GROUP RINGS

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Let R be a commutative ring having no nonzero nilpotent elements. We consider nil ideals in the group ring R[G]. In this paper conditions on G and R are found that ensure that Nil R[G], the upper nil radical of the group ring, is trivial. We also obtain necessary and sufficient conditions for the existence of nontrivial nilpotent ideals. In the particular case where R is a field K, the above results yield sufficient conditions on G and K for the semi-simplicity of the group algebra K[G]. These are similar to and, in fact, motivated by the results of S. A. Amitsur in [1] for fields of characteristic zero.

If there exists a positive integer $\bar{m}$ with $\bar{m}R = (0)$, then we set ch R, the characteristic of R, equal to the smallest such positive integer. Otherwise, ch R = 0. In the first case, let $\pi(R) = \{p_1, p_2, \cdots, p_k\}$ be the set of prime divisors of m = ch R. Then since R has no nontrivial nilpotent elements, it is immediate that each prime occurs only to the first power. The integers $m/p_1, m/p_2, \cdots, m/p_k$ are relatively prime, so there is a linear sum

$$n_1 m/p_1 + n_2 m/p_2 + \cdots + n_k m/p_k = 1 .$$

This induces a decomposition

$$R = R_{p_1} \dot{+} R_{p_2} \dot{+} \cdots \dot{+} R_{p_k}$$

of R as an internal direct sum of nonzero ideals, where

$$R_{p_i} = n_i m/p_i R = \{r \in R: p_i r = 0\} .$$

Hence, for any group G,

$$R[G] = R_{p_1}[G] \dot{+} R_{p_2}[G] \dot{+} \cdots \dot{+} R_{p_k}[G].$$

An ideal of the group ring is then nil or nilpotent if and only if its projection into each factor is. This effectively reduces all ch R ≠ 0 considerations to the prime case.

We say an element $\sigma \in G$ is a p element if it is of order p^j for some $j \geq 1$.

THEOREM I. *Let* R *be a commutative ring having no nonzero nilpotent elements. Suppose that* ch R ≠ 0 *and that* G *has no* p *elements for all* $p \in \pi(R)$. *Then* Nil R[G] = (0).

First, we need a few lemmas. Let Γ be any ring. We write Comm Γ for the commutator of Γ, the set of all finite sums of elements of the form ab - ba with a, b ∈ Γ.

LEMMA 1. *Let* p *be a prime, and let* k *and* n *be arbitrary positive integers. Then for all* $x_1, x_2, \cdots, x_n \in \Gamma$,

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$$(x_1 + x_2 + \cdots + x_n)^{p^k} = x_1^{p^k} + x_2^{p^k} + \cdots + x_n^{p^k} + p\gamma + z,$$

where $\gamma \in \Gamma$ *and* $z \in \mathrm{Comm}\,\Gamma$.

Proof. Observe that

$$(x_1 + x_2 + \cdots + x_n)^{p^k} = x_1^{p^k} + x_2^{p^k} + \cdots + x_n^{p^k} + t,$$

where t is the sum of all words of the form

$$x_{i_1} x_{i_2} \cdots x_{i_{p^k}}$$

with at least two different subscripts occurring.

If words w_1 and w_2 are cyclic permutations of each other, that is, if

$$w_1 = x_{i_1} x_{i_2} \cdots x_{i_{p^k}},$$

$$w_2 = x_{i_j} x_{i_{j+1}} \cdots x_{i_{p^k}} x_{i_1} \cdots x_{i_{j-1}},$$

then

$$w_1 - w_2 = ab - ba \in \mathrm{Comm}\,\Gamma, \quad \text{with}$$

$$a = x_{i_1} x_{i_2} \cdots x_{i_{j-1}}, \quad b = x_{i_j} x_{i_{j+1}} \cdots x_{i_{p^k}}.$$

Hence, modulo $\mathrm{Comm}\,\Gamma$ all cyclic permutations of a word w are congruent. But an easy counting argument shows that the number of such words is divisible by p. Thus the result follows.

For convenience we introduce the following notation. For all

$$a = \sum r_\sigma \sigma \in R[G],$$

we set $\theta(a) = r_1$, the coefficient of the identity of G.

Now $\mathrm{Comm}\,R[G]$ is spanned over R^2 by all elements of the form $\sigma\tau - \tau\sigma$ with $\sigma, \tau \in G$. But $\sigma\tau = 1$ if and only if $\tau\sigma = 1$, so we have proved

LEMMA 2. *If* $z \in \mathrm{Comm}\,R[G]$, *then* $\theta(z) = 0$.

LEMMA 3. *Let* R *be a commutative ring having no nonzero nilpotent elements, and let* $\mathrm{ch}\,R = p > 0$. *If*

$$a = r_1 1 + r_2\sigma_2 + \cdots + r_n\sigma_n \in R[G]$$

is nilpotent and no σ_i *is a* p *element, then* $\theta(a) = r_1 = 0$.

Proof. Suppose $a^{p^k} = 0$. Then

$$0 = a^{p^k} = r_1^{p^k} 1 + r_2^{p^k} \sigma_2^{p^k} + \cdots + r_n^{p^k} \sigma_n^{p^k} + pb + z$$

by Lemma 1. Now $pb = 0$ since $\mathrm{ch}\, R = p$ and, by assumption, no $\sigma_i^{p^k}$ is equal to 1. Hence,

$$r_1^{p^k} = -\theta(z) = 0$$

by Lemma 2. Therefore $r_1 = 0$.

Proof of theorem. It suffices to show that for each $p \in \pi(R)$, $\mathrm{Nil}\, R_p[G] = (0)$. But R_p has characteristic p, and G has no p elements. If

$$a = \sum r_\sigma \sigma \in \mathrm{Nil}\, R_p[G],$$

then for all $\tau \in G$, $a\tau^{-1}$ is nilpotent. Therefore by the preceeding lemma, $\theta(a\tau^{-1}) = r_\tau = 0$. Hence, $a = 0$.

For the case where ch R is zero, we have the following result.

THEOREM II. *Let* R *be a commutative ring without nonzero nilpotent elements. Suppose* $(R, +)$ *is torsion free. Then for any group* G, $\mathrm{Nil}\, R[G] = (0)$.

We first consider a special case.

LEMMA 4. *Let* K *be a finite field extension of the rational numbers* Q. *Then* $\mathrm{Nil}\, K[G] = (0)$.

Proof. Let D be the integral closure in K of the rational integers Z. Then K is the quotient field of D.

Suppose $\mathrm{Nil}\, K[G] \neq (0)$. Then by multiplying by a suitable group element and a field element, we can suppose

$$\alpha = d_1 1 + d_2 \sigma_2 + \cdots + d_n \sigma_n \in D[G]$$

is nilpotent and $d_1 \neq 0$.

The norm map $N_{K/Q}$ maps D into Z, and by assumption, $N_{K/Q}(d_1) \neq 0$. Choose a prime p with

$$\begin{aligned} p &> \text{absolute value of } N_{K/Q}(d_1), \\ &> \text{all the finite orders of the } \sigma_i, \\ &> \text{the degree of nilpotence of } \alpha. \end{aligned}$$

Then

$$0 = \alpha^p = d_1^p 1 + d_2^p \sigma_2^p + \cdots + d_n^p \sigma_n^p + p\beta + z,$$

where $\beta \in D[G]$ and $z \in \mathrm{Comm}\, D[G]$. Hence $d_1^p = pd$ with $d = -\theta(\beta)$ by Lemma 2.

Thus if $t = [K: Q]$,

$$N_{K/Q}(d_1)^p = N_{K/Q}(d_1^p) = N_{K/Q}(pd) = p^t N_{K/Q}(d).$$

But p does not divide $N_{K/Q}(d_1)$, so this is the required contradiction.

To reduce the general problem to this special case, we need the following two lemmas, whose proofs are immediately clear.

LEMMA 5. *Let* $R^{\#}$ *be the canonical ring extension of* R *having a unit element. Then* Nil $R[G] \subseteq$ Nil $R^{\#}[G]$.

LEMMA 6. *Let* S *be a multiplicative subset of regular elements (nonzero divisors) of* R. *Then* Nil $S^{-1}R[G] = S^{-1}$ Nil $R[G]$.

Proof of theorem. By Lemma 5, we can suppose that R contains a unit element. The hypotheses of the theorem state precisely that the nonzero integers Z_* in R form a multiplicative subset of regular elements. So by the preceeding lemma, we need only consider the ring $Z_*^{-1}R$, or in other words, we can assume that R is a unitary over-ring of the rationals Q. Note that neither of these extensions of R can introduce nonzero nilpotent elements in the ring.

The result now follows by the Hilbert Nullstellensatz. Let

$$\alpha = r_1\sigma_1 + r_2\sigma_2 + \cdots + r_n\sigma_n \in \text{Nil } R[G],$$

and let $\overline{R} = Q[r_1, r_2, \cdots, r_n] \subseteq R$. Then $\alpha \in \text{Nil } \overline{R}[G]$. Let M be a maximal ideal of $\overline{R}$. The image of α under the map

$$\overline{R}[G] \rightarrow \overline{R}/M[G]$$

is an element of Nil $\overline{R}/M[G]$. But $\overline{R}/M$ is a finite algebraic extension of Q, so by Lemma 4, the image is zero.

Hence, $r_1, r_2, \cdots, r_n \in M$ for all maximal ideals M. That is the r_i belong to the Jacobson radical of $\overline{R}$. But $\overline{R}$ is a finitely generated commutative algebra, and a second application of the Nullstellensatz implies that $r_1, r_2, \cdots, r_n$ are nilpotent. Finally, since R has no nonzero nilpotent elements, $\alpha = 0$.

Now, Theorem I yields sufficient conditions for the nil radical to be zero. That it need not be zero can be seen from the following example.

Let A be an abelian group having no elements of even order, and let $X = \{1, x\}$ be a group of order two. Set G equal to the semi-direct product of A by X, where x acts on A by sending each element to its inverse. We consider K[G] for K a field of characteristic two.

PROPOSITION. *With the above notation,*

(i) *if* A *is finite,* $\text{Nil } K[G] = K \cdot (\sum_{\rho \in G} \rho)$;

(ii) *if* A *is infinite,* $\text{Nil } K[G] = (0)$.

Proof. We use the notation $a, b \in K[A]$ so that any element of K[G] can be written uniquely as $a + bx$. If $a = \sum_{\sigma \in A} k_\sigma \sigma$, we set $a^{-1} = \sum_{\sigma \in A} k_\sigma \sigma^{-1}$. Notice that $ax = xa^{-1}$.

Now, if a is nilpotent, then since K[A] is commutative, a generates a nil ideal in K[A]. But by Theorem I, Nil $K[A] = (0)$, so in this case $a = 0$.

Let $a + bx \in \text{Nil } K[G]$, then

$$(a + bx)(a^{-1} + bx) = (aa^{-1} + bxbx) + (bxa^{-1} + abx),$$
$$= (aa^{-1} + bb^{-1}) + (ba + ab)x,$$
$$= (aa^{-1} + bb^{-1}) \in \text{Nil } K[G].$$

Thus $aa^{-1} = bb^{-1}$ since ch $K = 2$.

Therefore,

$$(a + bx)^2 = (aa + bxbx) + (abx + bxa),$$
$$= (aa + bb^{-1}) + (abx + a^{-1}bx),$$
$$= (aa + aa^{-1}) + (abx + a^{-1}bx),$$
$$= (a + a^{-1})(a + bx).$$

Thus if $(a + bx)^m = 0$,

$$0 = (a + a^{-1})^{m-1}(a + bx)$$

so that $(a + a^{-1})^{m-1}a = 0$. Moreover,

$$(a + a^{-1})^{m-1}a^{-1} = x[(a + a^{-1})^{m-1}a]x = 0.$$

Therefore, by addition, $(a + a^{-1})^m = 0$. Hence $a + a^{-1}$ is nilpotent, so $a = a^{-1}$.

Now if $\sigma \in A$, then

$$\sigma(a + bx) \in \text{Nil } K[G],$$

and $\sigma a = (\sigma a)^{-1}$. Suppose $a \neq 0$, and choose σ so that $\theta(\sigma a) \neq 0$. Then since 1 is the only element of A which is its own inverse, the number of nonzero terms of σa, and hence of a, is odd since $(\sigma a) = (\sigma a)^{-1}$. But if $\tau \in A$ and the coefficient of τ in a is zero, then $\theta(\tau^{-1}a) = 0$, which inplies that the number of nonzero terms is even. Therefore, all elements of A must appear.

Thus if A is infinite, $a = 0$ and $(bx)x = b \in \text{Nil } K[G]$. Consequently, $b = 0$. Therefore (ii) is proved.

Now assume A is finite and thus is of odd order. Then every element of A has a square root. Suppose $a = \sum_{\sigma \in A} k_\sigma \sigma$. Fix $\sigma_1, \sigma_2 \in A$, and choose τ so that $\tau^2 = \sigma_1^{-1}\sigma_2^{-1}$. Then $\tau\sigma_1 = (\tau\sigma_2)^{-1}$. But $(\tau a) = (\tau a)^{-1}$, so $k_{\sigma_1} = k_{\sigma_2}$. Hence $a = k \sum_{\sigma \in A} \sigma$.

We also see that

$$(a + bx)x = b + ax \quad \text{in Nil } K[G],$$

so we obtain similarly the result that $b = \bar{k} \sum_{\sigma \in A} \sigma$. Clearly,

$$z = \bar{k} \sum_{\rho \in G} \rho \in \mathrm{Nil}\ K[G];$$

therefore,

$$(a + bx) - z = (k - \bar{k}) \sum_{\sigma \in A} \sigma$$

is nilpotent. Hence $k = \bar{k}$, and (i) is proved.

Necessary and sufficient conditions for the existence of nontrivial nil ideals in R[G] are not immediately apparent. However we have found them for nilpotent ideals.

THEOREM III. *Let* R *be a commutative ring having no nonzero nilpotent elements and such that* $\mathrm{ch}\ R \neq 0$. *Then* R[G] *has a nontrivial nilpotent ideal if and only if* G *contains a finite normal subgroup* H *whose order is divisible by some* $p \in \pi(R)$.

LEMMA 7. *Let* J *be a group, and let* $H_1, H_2, \cdots, H_n$ *be a finite number of its subgroups. Suppose there exists a finite collection of elements* $\sigma_{ij} \in J$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, f(i))$ *with*

$$J = \bigcup_{i,j} H_j \sigma_{ij} \qquad \text{(The union is set theoretic.)}$$

Then for some index i, $[J: H_i] < \infty$.

Proof. By relabeling, we can assume all the H_i to be distinct. We prove the result by induction on n, the number of distinct H_i. The case $n = 1$ is clear.

If a full set of cosets of H_n appears among the $H_n \sigma_{nj}$, then $[J: H_n] < \infty$, and we are finished. Otherwise, if $H_n \tau$ is a missing coset, then

$$H_n \tau \subseteq \bigcup_{i,j} H_i \sigma_{ij}.$$

But $H_n \tau \cap H_n \sigma_{nj}$ is empty, so

$$H_n \tau \subseteq \bigcup_{\substack{i \neq n \\ j}} H_i \sigma_{ij}.$$

Thus all the cosets $H_n \sigma_{nj}$ can be replaced by finite unions of cosets of the remaining H_i. Doing this yields a representation of J in terms of a smaller number of subgroups, and the result follows.

For any group G set

$$\Delta = \{\sigma \in G: \sigma \text{ has only a finite number of conjugates in } G\}.$$

If σ_1, σ_2 lie in Δ, then so does $\sigma_1 \sigma_2^{-1}$, since for any $\tau \in G$

$$\tau^{-1}(\sigma_1 \sigma_2^{-1})\tau = [\tau^{-1}\sigma_1 \tau][\tau^{-1}\sigma_2 \tau]^{-1},$$

and there are only a finite number of possibilities for each factor. Thus $\triangle$ is a subgroup of G, and it is clearly characteristic.

We denote by ψ the projection of $R[G]$ into $R[\triangle]$, that is,

$$\psi\colon R[G] \to R[\triangle].$$

If $\alpha = \Sigma r_\sigma \sigma \in R[G]$, then

$$\psi(\alpha) = \sum_{\sigma \in \triangle} r_\sigma \sigma \in R[\triangle].$$

For convenience, given $\alpha = \Sigma r_\sigma \sigma \in R[G]$, we write

$$\text{Supp}\,\alpha = \{\sigma \in G\colon r_\sigma \neq 0\},$$

which is the support of α.

LEMMA 8. *Let* $\alpha \in R[G]$, *and suppose that for all* $\sigma \in G$

$$\psi(\sigma^{-1}\alpha\sigma\alpha) = 0.$$

Then if $\alpha_0 = \psi(\alpha)$, $\alpha_0^2 = 0$.

Proof. Write $\alpha = \alpha_0 + \beta$, where

$$\beta = r_1\tau_1 + \cdots + r_n\tau_n \qquad (\tau_i \notin \triangle).$$

For each $\rho \in \text{Supp}\,\alpha_0$, $[G\colon C(\rho)] < \infty$, $C(\rho)$ being the centralizer of ρ. Let

$$J = \bigcap_{\rho \in \text{Supp}\,\alpha_0} C(\rho).$$

Then $[G\colon J] < \infty$; and for all $\sigma \in J$, $\sigma^{-1}\alpha_0\sigma = \alpha_0$. Set $H_i = J \cap C(\tau_i)$ $(i = 1, 2, \cdots, n)$.

Assume $\alpha_0^2 \neq 0$, and choose a $\xi \in \text{Supp}\,\alpha_0^2$. For each i, if τ_i is conjugate to $\xi\tau_j^{-1}$ by an element of J, choose $\sigma_{ij} \in J$ so that

$$\sigma_{ij}^{-1}\tau_i\sigma_{ij} = \xi\tau_j^{-1}.$$

Let $\sigma \in J$. Then

$$\sigma^{-1}\alpha\sigma\alpha = (\alpha_0 + \sigma^{-1}\beta\sigma)(\alpha_0 + \beta),$$

$$= \alpha_0^2 + \sigma^{-1}\beta\sigma\alpha_0 + \alpha_0\beta + \sigma^{-1}\beta\sigma\beta.$$

Now $\psi(\sigma^{-1}\alpha\sigma\alpha) = 0$. Hence the ξ term of α_0^2 must be cancelled out by a ξ term in one of the remaining three summands. Thus it is clear, since

$$\text{Supp}(\sigma^{-1}\beta\sigma\alpha_0) \cap \triangle = \text{Supp}(\alpha_0\beta) \cap \triangle = \emptyset \text{ and } \xi \in \triangle,$$

that

$$\xi \in \text{Supp}(\sigma^{-1}\beta\sigma\beta).$$

Consequently there exist τ_i, τ_j with

$$\sigma^{-1}\tau_i\sigma\tau_j = \xi$$

or

$$\sigma^{-1}\tau_i\sigma = \xi\tau_j^{-1}.$$

But this means that $\sigma \in H_i\sigma_{ij}$.

Therefore we have shown that

$$J = \bigcup_{i,j} H_i\sigma_{ij}.$$

Hence by the preceeding lemma, for some index i, $[J: H_i] < \infty$. But $H_i \subseteq C(\tau_i)$, so $[G: C(\tau_i)] < \infty$. This is the required contradiction since $\tau_i \notin \Delta$.

LEMMA 9. *Let* $S = \{x_1, x_2, \cdots, x_n\}$ *be a finite normal subset of* G. *If* $H = \langle S \rangle$ *the subgroup generated by* S *then* H *is normal in* G *and for all* $\sigma \in H$,

$$\sigma = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

for some integers $m_1, m_2, \cdots, m_n$. *In particular, if each* x_i *is of finite order, then so is* H.

Proof. Clearly, H is normal.

Any element of H can be written in the form

$$w = x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \cdots x_{i_t}^{\pm 1},$$

and we prove the result by induction on the length t of the word. For $t = 1$ there is nothing to prove.

Consider the smallest subscript occurring in the representation of w. Name it j. Then

$$w = w_1 x_j^{\pm 1} w_2 = x_j^{\pm 1}(x_j^{\mp 1} w_1 x_j^{\pm 1}) w_2 = x_j^{\pm 1} w_3,$$

since S is normal.

But w_3 has one less element, so we can write

$$w_3 = x_k^{m_k} x_{k+1}^{m_{k+1}} \cdots x_n^{m_n},$$

omitting the first terms with zero exponent.

If $j \leq k$, we are through. Otherwise, there exists an element of smaller subscript in this representation of w, and we apply this process again. Clearly this procedure terminates after a finite number of applications since S is finite, and the desired result follows.

Proof of Theorem III. Suppose H is a finite normal subgroup of G with order divisible by p for some $p \in \pi(R)$. Choose a nonzero $r \in R_p$, and set

$$\alpha = \sum_{h \in H} rh \in R[G].$$

Then α is nilpotent and central in $R[G]$, so it generates a nilpotent ideal.

Conversely, let N be a nontrivial nilpotent ideal of $R[G]$. Suppose

$$N^m \neq (0), \quad N^{m+1} = (0).$$

For some $p \in \pi(R)$, the projection of N^m into $R_p[G]$ is nonzero. We choose α in this image with $\theta(\alpha) \neq 0$.

Then for all $\sigma \in G$,

$$\sigma^{-1}\alpha\sigma\alpha = 0,$$

so by Lemma 8, $\alpha_0 = \psi(\alpha)$ is nilpotent. But $\theta(\alpha_0) = \theta(\alpha) \neq 0$. Thus by Lemma 3, Supp α_0 contains a p element λ. Now $\lambda \in \Delta$, and if we let S be the finite set of conjugates of λ, then S is normal. By the preceeding lemma, $H = \langle S \rangle$ is a finite normal subgroup of G with order divisible by p, since $\lambda \in H$. This completes the proof.

We now consider the question of the semi-simplicity of the group algebra $K[G]$. We let Rad Γ denote the Jacobson radical of any ring Γ.

THEOREM IV. *Let* K *be a field of characteristic* $p > 0$, *and let* G *have no elements of order* p. *If* K *is a separably generated non-algebraic extension of some subfield* K_0, *then* $K[G]$ *is semi-simple.*

THEOREM V. (Amitsur [1]) *Let* K *be a field of characteristic zero. If* K *is a non-algebraic extension of the rational numbers, then for all groups* G, $K[G]$ *is semi-simple.*

Proof. We consider the first result. Let $(x) = (x_\mu)$ be a transcendence base for K over K_0. This base is not assumed to be finite but just non-empty. Set $K_1 = K_0(x)$. Then by Theorem II of [2],

$$\text{Rad } K_1[G] \subseteq K_1 \otimes_{K_0} \text{Nil } K_0[G],$$

and since the nil radical is zero by Theorem I above, $K_1[G]$ is semi-simple.

But now K is a separable algebraic extension of K_1, and by Theorem I of [2],

$$\text{Rad } K[G] = K \otimes_{K_1} \text{Rad } K_1[G].$$

So Rad $K[G] = 0$, and $K[G]$ is semi-simple.

Theorem V follows similarly.

The latter theorem asserts, in particular, the semi-simplicity of all group algebras over nondenumerable fields of characteristic zero. However the former one does not yield a corresponding result for characteristic $p > 0$ because of the separability condition. But this result can be obtained as follows.

LEMMA 10. *Let* H *be a subgroup of* G, *and let* $a \in K[H]$. *If* a *is quasi-regular in* $K[G]$, *then it is in* $K[H]$.

Proof. This follows immediately because the projection of any quasi-inverse of a into K[H] is also a quasi-inverse of a.

THEOREM VI. *Let* K *be a nondenumerable field of characteristic* $p > 0$, *and let* G *be a group having no elements of order* p. *Then* K[G] *is semi-simple.*

Proof. Let

$$a = \sum k_\sigma \sigma \in \text{Rad } K[G],$$

and let H be the subgroup of G generated by Supp a. Then H is finitely generated, and $a \in K[H]$. If b is any element of K[H], then ba is quasi-regular in K[G], and hence in K[H], by the preceeding lemma. Thus $a \in$ Rad K[H]. But K[H] is a finitely generated algebra over a nondenumerable field. Thus, by Corollary 4 of [3], $a \in$ Nil K[H]. By Theorem I, the nil radical is zero, and thus $a = 0$.

Therefore with the assumption that G has no elements of order p, the basic results of [1] carry over to fields of characteristic $p > 0$.

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