DIFFERENTIAL SYSTEMS AND EXTENSION OF LYAPUNOV'S METHOD

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Let I denote the half-line $0 \leq t < \infty$, and let R^n denote n-dimensional Euclidian space. We consider the differential systems

$$\left.\begin{array}{ll} (1) & x' = f(t, x);\ \ x(t_0) = x_0, \\ (2) & y' = g(t, y);\ \ y(t_0) = y_0, \end{array}\right\} \quad (t_0 \geq 0)$$

where x, y, f and g are n-dimensional vectors, and where the functions f(t, x), g(t, y) are defined and continuous on the product space $I \times R^n$. In Theorems 1 to 11 below, we establish a number of results on the stability and boundedness of solutions of the systems (1) and (2). Our results constitute an extension of work of Yoshizawa [7], [8], Brauer [1], and Conti [2].

We adopt the notation $R^+ = [0, \infty)$ and $|x| = \Sigma_{i=1}^n |x_i|$, and we shall write d(x, y) for $|x - y|$. Let a function $V(t, x, y) \geq 0$ be defined and continuous on the product space $I \times R^n \times R^n$, and suppose that it satisfies Lipschitz's condition in x and y locally. In particular, we assume that $V(t, x, x) \geq 0$ for (t, x) in $I \times R^n$. Following Yoshizawa [7], we next define the function

$$(3) \qquad V^*(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + hf(t, x), y + hg(t, y)) - V(t, x, y)].$$

With respect to these functions we state the following lemmas.

LEMMA 1. *Let the function* W(t, r) *be defined and continuous on* $I \times R^+$. *Suppose further that the function* $V^*(t, x, y)$ *of* (3) *satisfies the condition*

$$(4) \qquad V^*(t, x, y) \leq W(t, V(t, x, y)).$$

Let r(t) *be the maximum solution of the differential equation*

$$(5) \qquad r' = W(t, r), \qquad r(t_0) = r_0 \geq 0.$$

If x(t) *and* y(t) *are any two solutions of* (1) *and* (2) *such that* $V(t_0, x_0, y_0) \leq r_0$, *then*

$$(6) \qquad V(t, x(t), y(t)) \leq r(t) \qquad (t \geq t_0).$$

LEMMA 2. *If the assumptions of Lemma* 1 *hold, except that the condition* (4) *is replaced by the inequality*

$$(7) \qquad A(t)V^*(t, x, y) + A^*(t)V(t, x, y) \leq W(t, A(t)V(t, x, y)),$$

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where the function $A(t)$ *is continuous, positive and defined on* I, *and*

$$A^*(t) = \limsup_{h\to 0^+} [A(t+h) - A(t)]/h,$$

and if $A(t_0)V(t_0, x_0, y_0) \le r_0$, *then*

$$A(t)V(t, x(t), y(t)) \le r(t) \qquad (t \ge t_0). \tag{8}$$

Remark. Taking $A(t) \equiv 1$, we see that Lemma 2 reduces to Lemma 1. Since Lemma 1 is an important tool by itself in the study of various problems of differential equations, we have stated it separately. Actually, it is an extension of a lemma due to Conti [2], who assumes that $V(t, x, y)$ has continuous partial derivatives with respect to t and with respect to the components of x and y. F. Brauer improves Conti's lemma, assuming one-sided partial derivatives [1].

Proof of Lemma 1. Let $x(t)$ and $y(t)$ be any two solutions of (1) and (2) such that $V(t_0, x_0, y_0) \le r_0$. Define $m(t) = V(t, x(t), y(t))$. Then, using the hypothesis that $V(t, x, y)$ satisfies Lipschitz's condition with respect to x and y, we obtain, for small, positive h, the inequality

$$\begin{aligned} m(t+h) - m(t) \le K[|\varepsilon_1 h| + |\varepsilon_2 h|] \\ + V[t+h, x(t) + hf(t, x(t)), y(t) + hg(t, y(t))] - V(t, x(t), y(t)), \end{aligned}$$

where K is the Lipschitz constant at (t, x, y) and ε_1 and ε_2 tend to zero as h tends to zero. This together with condition (4) yields the inequality

$$\limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \le W(t, m(t)).$$

The standard argument used in [3], [4] can now be followed to establish the result (6).

Proof of Lemma 2. Let $R(t)$ be the maximum solution of

$$R' = [-A^*(t)R + W(t, A(t)R)]A^{-1}(t), \qquad R(t_0) \ge V(t_0, x_0, y_0).$$

Then it is clear from (7) that Lemma 1 can be used to obtain the inequality

$$V(t, x(t), y(t)) \le R(t) \qquad (t \ge t_0).$$

But $R(t) = r(t)A^{-1}(t)$, where $r(t)$ is the maximal solution of (5) such that $R(t_0) = r_0 A^{-1}(t_0)$. Hence the result follows.

Remark. Lemma 2 can also be reduced to Lemma 1 by defining

$$L(t, x, y) = A(t)V(t, x, y)$$

and verifying that $L(t, x, y)$ preserves the properties of $V(t, x, y)$. The former proof was also pointed out to us by F. Brauer.

We assume hereafter that the solutions $r(t)$ of (5) are nonnegative for $t \ge t_0$ so as to ensure that $W(t, r(t))$ is defined. Such a requirement is clearly satisfied if we assume that $W(t, 0) = 0$ for all t.

THEOREM 1. *Let the assumptions of Lemma* 1 *hold. Suppose also that* $V(t, x, y) \to \infty$ *as* $d(x, y) \to \infty$. *Then, if all solutions of* (5) *together with one solution of* (2) *can be continued for all* t, *all the solutions of* (1) *can be continued for all* t.

Suppose that $g(t, y) \equiv 0$. In this case, the result is a refinement of the global existence theorem of Conti [2] and Brauer [1]. The proof of the theorem is immediate (one uses a result of Wintner [6] as in [1]).

For any fixed (t_0, y_0), let $y(t)$ be any solution of (2) which exists in $[t_0, \infty)$. If there exists a sequence $\{t_k\}$ $(t_k \to \infty)$ such that $y(t_k) \to \mu \in R^n$ as $k \to \infty$, we say that μ is a *cluster point* of $y(t)$. We denote by M the set of cluster points of $y(t)$.

THEOREM 2. *Let the assumptions of Lemma* 2 *hold. Suppose* $A(t) \to \infty$ *as* $t \to \infty$. *Suppose that* (α)

$$b(d(x, y)) \leq V(t, x, y), \tag{9}$$

where the function $b(r)$ *is continuous and nondecreasing in* r, *and* $b(r) > 0$ *for* $r > 0$, (β) M *is the cluster set of* $y(t)$, *and* (γ) *all the solutions of* (5) *remain bounded as* $t \to \infty$. *Then* M *is the cluster set of every solution* $x(t)$ *of* (1).

Proof. Since M is the cluster set of $y(t)$, it is enough to prove that $d(x(t), y(t)) \to 0$ as $t \to \infty$. If $x(t)$ and $y(t)$ are any two solutions of (1) and (2) such that $A(t_0)V(t_0, x_0, y_0) \leq r_0$, we deduce from Lemma 2 that

$$A(t)V(t, x(t), y(t)) \leq r(t) \qquad (t \geq t_0). \tag{10}$$

Let $\{t_k\}$ $(t_k \to \infty)$ be a sequence such that $|x(t_k) - y(t_k)| \geq \varepsilon$, for some $\varepsilon > 0$ and for each k. Then it follows from (9) and (10) that

$$A(t_k)b(\varepsilon) \leq A(t_k)V(t_k, x(t_k), y(t_k)) \leq r(t_k) \leq B,$$

since all the solutions $r(t)$ of (5) are assumed to be bounded as $t \to \infty$. Since $b(\varepsilon) > 0$ and since $A(t_k) \to \infty$ as $t_k \to \infty$, there is a contradiction, which implies that M is the cluster set of $x(t)$. Hence the proof is complete.

Suppose that $x(t)$ and $y(t)$ are any two solutions of (1) and (2). In order to unify our results on stability and boundedness, we list the following conditions.

(i) For each $\alpha > 0$ and $t_0 \geq 0$, there exists a positive function $\beta(t_0, \alpha)$ that is continuous in t_0 for each α and such that if $d(x_0, y_0) \leq \alpha$ and $t \geq t_0$, then

$$d(x(t), y(t)) < \beta(t_0, \alpha).$$

(ii) The β in (i) is independent of t_0.

(iii) For each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists a positive function $\eta(t_0, \varepsilon)$ that is continuous in t_0 for each ε and such that if $d(x_0, y_0) \leq \eta(t_0, \varepsilon)$ then $d(x(t), y(t)) < \varepsilon$ for all $t \geq t_0$.

(iv) The η in (iii) is independent of t_0.

(v) For each $\alpha > 0$ and each $t_0 \geq 0$, there exist positive numbers B and $T(t_0, \alpha)$ such that $d(x(t), y(t)) < B$ provided $d(x_0, y_0) \leq \alpha$ and $t > t_0 + T(t_0, \alpha)$.

(vi) The T in (v) is independent of t_0.

(vii) (i) and (v) hold simultaneously.

(viii) (ii) and (vi) hold simultaneously.

(ix) For each $\varepsilon > 0$, $\alpha > 0$ and $t_0 \geq 0$, there exists a positive number $T(t_0, \varepsilon, \alpha)$ such that $d(x(t), y(t)) < \varepsilon$ provided $d(x_0, y_0) \leq \alpha$ and $t > t_0 + T(t_0, \varepsilon, \alpha)$.

(x) The T in (ix) is independent of t_0.

(xi) The conditions (iii) and (ix) hold simultaneously.

(xii) The conditions (iv) and (x) hold simultaneously.

Remark. Corresponding to the definitions above, if we say that the differential equation (5) has the property (ia), we mean that the following condition is satisfied:

(ia) Given $\alpha > 0$ and $t_0 \geq 0$, there exists a positive function $\beta(t_0, \alpha)$ that is continuous in t_0 for each α and that satisfies the inequality $r(t) < \beta(t_0, \alpha)$ if $r_0 \leq \alpha$ and $t \geq t_0$.

Conditions (ii) to (xii) may be reformulated similarly.

The following theorems on stability and boundedness are extensions of many results of Yoshizawa [7, 8]. We assume that

(11) the function $b(r)$ is continuous and nondecreasing in r, $b(r) > 0$ for $r > 0$,

and $b(d(x, y)) \leq V(t, x, y)$. On occasion, we may also assume, below, that

$$b(r) \to \infty \quad \text{as } r \to \infty . \tag{12}$$

THEOREM 3. *Let the assumptions of Lemma* 1 *hold, together with* (11) *and* (12). *Suppose further that the differential equation* (5) *satisfies one of the conditions* (ia), (iia), (va), (via), (viia), *and* (viiia); *then the systems* (1) *and* (2) *satisfy the corresponding one of the conditions* (i), (ii), (v), (vi), (vii), *and* (viii).

Proof. Suppose the differential equation (5) has property (ia). Then, corresponding to $\alpha > 0$ and $t_0 \geq 0$, there exists a positive function $\beta(t_0, \alpha)$, that is continuous in t_0 for each α and satisfies

$$r(t) < \beta(t_0, \alpha) \tag{13}$$

if $r_0 \leq \alpha$ and $t_0 \geq 0$. Since $b(r) \to \infty$ as $r \to \infty$, there exists an $L = L(t_0, \alpha)$ such that

$$b(L) > \beta(t_0, \alpha) . \tag{14}$$

Let $x(t)$ and $y(t)$ be any two solutions of (1) and (2) for which $V(t_0, x_0, y_0) \leq r_0 \leq \alpha$. Then it follows from (11) that

$$b(d(x_0, y_0)) \leq V(t_0, x_0, y_0) \leq \alpha .$$

Since $b(r)$ is nonnegative and increasing, $d(x_0, y_0) \leq b^{-1}(\alpha) \equiv \gamma$. Also, by Lemma 1,

$$V(t, x(t), y(t)) \leq r(t) \qquad \text{for } t \geq t_0 . \tag{15}$$

Now assume that there exist two solutions $x(t)$, $y(t)$ of (1) and (2) for which $d(x_0, y_0) \leq \gamma$ have the property that $d(x(t_1), y(t_1)) = L$ for some $t = t_1 > t_0$. Then from the relations (11), (13), (14), and (15), we obtain the inequality

$$b(L) \leq V(t_1, x(t_1), y(t_1)) \leq r(t_1) < \beta(t_0, \alpha) < b(L),$$

which is a contradiction. It therefore follows that if $d(x_0, y_0) \leq \gamma$ and $t \geq t_0$, then $d(x(t), y(t)) < L(t_0, \alpha)$. This proves (i).

The proof of (ii) is essentially the same, since $\beta(t_0, \alpha)$ is independent of t_0 in this case.

The proofs of the other statements are also similar. We shall only indicate the proof of the conclusion (v). Since equation (5) satisfies condition (va), given $\alpha > 0$ and $t_0 \geq 0$, there exist positive numbers B and $T(t_0, \alpha)$ such that

$$r(t) < B \tag{16}$$

for all $r_0 \leq \alpha$ and $t > t_0 + T(t_0, \alpha)$. Since $b(r) \to \infty$ as $r \to \infty$, there exists an M such that

$$b(M) > B. \tag{17}$$

Let x(t) and y(t) be any two solutions of (1) and (2) for which $V(t_0, x_0, y_0) \leq r_0 \leq \alpha$. Then, as in the previous case, $d(x_0, y_0) \leq \gamma$. If there exist two solutions x(t), y(t) of (1) and (2) that satisfy the condition $d(x_0, y_0) \leq \gamma$ and such that $d(x(t_k), y(t_k)) \geq M$ for some sequence $\{t_k\}$ $(t_k \to \infty)$, then, as in the proof of (i), the relations (15), (16) and (17) imply the inequality

$$b(M) \leq V(t_k, x(t_k), y(t_k)) \leq r(t_k) < B < b(M).$$

This contradiction shows that the systems (1) and (2) satisfy (v), and this completes the proof.

THEOREM 4. *Let the assumptions of Lemma* 1 *hold, together with* (11). *Suppose further that the differential equation* (5) *satisfies one of the conditions* (iiia), (iva), (ixa), (xa), (xia) *and* (xiia); *then the systems* (1) *and* (2) *satisfy the corresponding one of the conditions* (iii), (iv), (ix), (x), (xi) *and* (xii).

Proof. For each $\varepsilon > 0$, if $d(x, y) = \varepsilon$, we deduce from (11) that $b(\varepsilon) \leq V(t, x, y)$. If the equation (5) has property (iiia), given $b(\varepsilon) > 0$ and $t_0 \geq 0$, there exists a positive function $\eta(t_0, \varepsilon)$ such that

$$r(t) < b(\varepsilon) \tag{18}$$

if $r_0 \leq \eta(t_0, \varepsilon)$ and $t > t_0$. Suppose x(t) and y(t) are any two solutions of (1) and (2) such that $V(t_0, x_0, y_0) \leq r_0 \leq \eta(t_0, \varepsilon)$. By (11) and the monotonicity of b(r), this implies that

$$d(x_0, y_0) \leq b^{-1}(\eta(t_0, \varepsilon)) \equiv \delta(t_0, \varepsilon).$$

If we now assume that there exist two solutions x(t) and y(t) of (1) and (2) for which $d(x_0, y_0) \leq \delta(t_0, \varepsilon)$ and $d(x(t_1), y(t_1)) = \varepsilon$ for some $t = t_1 > t_0$, then, using Lemma 1 and the relations (11) and (18), we are led to the contradiction

$$b(\varepsilon) \leq V(t_1, x(t_1), y(t_1)) \leq r(t_1) < b(\varepsilon).$$

Therefore the systems (1) and (2) fulfill condition (iii).

By following the proof of Theorem 3 and that given above, we can easily construct proofs of the remaining conclusions of the theorem. We omit the details.

THEOREM 5. *Let the assumptions of Lemma* 2 *hold, together with* (11) *and* (12). *Suppose* $A(t) \to \infty$ *as* $t \to \infty$, *and suppose further that the differential equation* (5) *satisfies one of the conditions* (ia) *and* (iia). *Then the systems* (1) *and* (2) *satisfy the corresponding one of the conditions* (v) *and* (vi). *If in addition* $A(t) \geq 1$, *then the systems* (1) *and* (2) *have the properties* (vii) *and* (viii), *respectively.*

Proof. We first show that (v) is implied by (ia). Let x(t) and y(t) be any two solutions of (1) and (2) such that $A(t_0)V(t_0, x_0, y_0) \leq r_0$. Then it follows from Lemma 2 that

$$A(t)V(t, x(t), y(t)) \leq r(t) \quad \text{for } t \geq t_0. \tag{19}$$

Since the equation (5) satisfies (ia), given $\alpha > 0$ and $t_0 \geq 0$, there exists a positive number $\beta(t_0, \alpha)$ such that $r(t) < \beta(t_0, \alpha)$ if $r_0 \leq \alpha$. Since $b(r) \to \infty$ as $r \to \infty$, there exists an L such that

$$b(L) > \beta(t_0, \alpha). \tag{20}$$

Now, choosing $r_0 \leq \alpha$, we obtain the inequality

$$d(x_0, y_0) \leq b^{-1}(\alpha A^{-1}(t_0)) \equiv \gamma.$$

Suppose that there exist two solutions x(t) and y(t) of (1) and (2) satisfying the condition $d(x_0, y_0) \leq \gamma$ and such that $d(x(t_k), y(t_k)) \geq L$, where $t_k \to \infty$. Then it follows from (11), (19), and (20) that

$$A(t_k)b(L) \leq A(t_k)V(t_k, x(t_k), y(t_k)) \leq r(t_k) < \beta(t_0, \alpha) < b(L).$$

Since $A(t_k) \to \infty$ as $t_k \to \infty$ and since $b(L) > 0$, this implies a contradiction; hence the systems (1) and (2) satisfy (v).

If $A(t) \geq 1$, then, following the proof of Theorem 3, we also find that the systems (1) and (2) have property (i). Hence they have property (vii). Similar conclusions hold for the other case. The proof is complete.

THEOREM 6. *Let the assumptions of Lemma* 2 *hold, together with* (11). *Suppose* $A(t) \to \infty$ *as* $b \to \infty$, *and suppose further that the differential equation* (5) *satisfies one of the conditions* (iiia) *and* (iva). *Then the systems* (1) *and* (2) *satisfy the corresponding one of the conditions* (ix) *and* (x). *If in addition* $A(t) \geq 1$, *they also have the corresponding one of properties* (xi) *and* (xii).

Proof. For any $\varepsilon > 0$, if $d(x, y) = \varepsilon$, then by (11), $b(\varepsilon) \leq V(t, x, y)$. If equation (5) satisfies (iiia), then given $b(\varepsilon) > 0$ and $t_0 \geq 0$, there exists a positive number $\eta(t_0, \varepsilon)$ such that

$$r(t) < b(\varepsilon) \tag{21}$$

if $r_0 \leq \eta(t_0, \varepsilon)$ and $t \geq t_0$. Let x(t) and y(t) be any two solutions of (1) and (2) such that

$$A(t_0)V(t_0, x_0, y_0) \leq r_0 \leq \eta(t_0, \varepsilon).$$

In view of (11), this means that

$$b(d(x_0, y_0)) \leq \eta(t_0, \varepsilon) A^{-1}(t_0),$$

and by the monotonic property of $b(r)$,

$$d(x_0, y_0) \leq b^{-1}(\eta(t_0, \varepsilon) A^{-1}(t_0)) \equiv \alpha .$$

Now if there exist two solutions $x(t)$ and $y(t)$ of (1) and (2) for which $d(x_0, y_0) \leq \alpha$ and $d(x(t_k), y(t_k)) \geq \varepsilon$ for some sequence $\{t_k\}$ $(t_k \to \infty)$, then it follows from (11), (19) and (21) that

$$A(t_k) b(\varepsilon) \leq A(t_k) V(t_k, x(t_k), y(t_k)) \leq r(t_k) < b(\varepsilon) .$$

This leads to a contradiction, since $A(t_k) \to \infty$ as $t_k \to \infty$ and $b(\varepsilon) > 0$. Hence it follows that the systems (1) and (2) satisfy (ix). If in addition $A(t) \geq 1$, then, in analogy to the proof of Theorem 4, we find that the systems (1) and (2) satisfy (iii). This implies that they satisfy (xi). The proof in the other case is similar. We leave the details to the reader.

We now extend the preceeding results to perturbed systems. Corresponding to (1) and (2), we consider the systems

$$\left.\begin{array}{lll} (1^*) & x' = f(t, x) + F(t, x), & x(t_0) = x_0, \\ (2^*) & y' = g(t, y) + G(t, y), & y(t_0) = y_0, \end{array}\right\} \quad (t_0 \geq 0) .$$

If the solutions of (1*) and (2*) satisfy the conditions (i) to (xii) for all the perturbations F and G for which

$$|F(t, x)| + |G(t, y)| \leq \eta V(t, x, y) \qquad (\eta > 0), \tag{22}$$

we say that the systems (1) and (2) satisfy the definition (i) to (xii) *weakly.*

The following analogous theorems for weak boundedness and stability may be stated.

THEOREM 7. *Let the assumptions of Lemma* 1 *hold, except that the condition* (4) *is replaced by*

$$V^*(t, x, y) + \alpha V(t, x, y) \leq W(t, V(t, x, y)), \tag{23}$$

where $\alpha = K\eta$ (K *is the Lipschitz constant at* (t, x, y)). *Suppose also that the hypotheses* (11), *and* (12) *are fulfilled. Then, if the differential equation* (5) *satisfies one of the conditions* (ia), (iia), (va), (via), (viia), *and* (viiia), *then the systems* (1) *and* (2) *satisfy weakly the corresponding one of conditions* (i), (ii), (v), (vi), (vii) *and* (viii).

THEOREM 8. *Let the assumptions in the first sentence of Theorem* 7 *hold, together with* (11). *If the differential equation* (5) *satisfies one of the conditions* (iiia), (iva), (ixa), (xa), (xia) *and* (xiia), *then the systems* (1) *and* (2) *satisfy weakly the corresponding one of conditions* (iii), (iv), (ix), (x), (xi) *and* (xii).

Proof of Theorems 7 *and* 8. Since $V(t, x, y)$ satisfies Lipschitz's condition in both x and y,

$$\begin{aligned} &V[t + h, x + h(f(t, x) + F(t, x)), y + h(g(t, y) + G(t, y))] - V(t, x, y) \\ (24)\qquad &\leq K[|F(t, x)| + |(G(t, y)|]h \\ &+ V(t + h, x + hf(t, x), y + hg(t, y)) - V(t, x, y), \end{aligned}$$

for h positive and sufficiently small. Now, using (22) and (23) and noting that $\alpha = K\eta$, we obtain the inequality

$$V^{**}(t, x, y) \leq W(t, V(t, x, y)),$$

where

$$V^{**}(t, x, y)$$

$$= \limsup_{h \to 0^+} \frac{1}{h}(V[t + h, x + h(f(t, x) + F(t, x)), y + h(g(t, y) + G(t, y))] - V(t, x, y)).$$

If x(t) and y(t) are any two solutions of (1*) and (2*), we can obtain the desired results by applying directly the proofs of Lemma 1 and Theorems 3 and 4.

THEOREM 9. *Suppose that the assumptions of Lemma* 1 *hold, except that the condition* (4) *is replaced by*

$$V^*(t, x, y) + \alpha V(t, x, y) \leq W(t, V(t, x, y)e^{\beta t})e^{-\beta t}, \tag{25}$$

where β is positive and satisfies the inequality $\alpha \geq K\eta + \beta$. Let the assumptions (11) *and* (12) *hold. Then, if the differential equation* (5) *satisfies the conditions* (ia) *and* (iia), *respectively, the systems* (1) *and* (2) *satisfy weakly the conditions* (vii) *and* (viii), *respectively.*

THEOREM 10. *Let the assumptions of Theorem* 9 *hold except for* (12), (ia) *and* (iia). *Let the differential equation* (5) *satisfy condition* (iiia) *or* (iva), *respectively; then the systems* (1) *and* (2) *satisfy weakly definition* (xi) *or* (xii), *respectively.*

Proof of Theorems 9 *and* 10. Proceeding as in the proof of Theorems 7 and 8, we obtain the inequality

$$V^{**}(t, x, y) + \beta V(t, x, y) \leq W(t, V(t, x, y)e^{\beta t})e^{-\beta t}.$$

This is similar to condition (7) of Lemma 2 with $A(t) = e^{\beta t}$. Hence, using Lemma 2, we find that

$$V(t, x(t), y(t))e^{\beta t} \leq r(t) \qquad (t \geq t_0),$$

where x(t) and y(t) are any two solutions of (1*) and (2*) such that

$$e^{\beta t_0} V(t_0, x_0, y_0) \leq r_0.$$

Now, following the proofs of Theorems 5 and 6, we can establish the theorems.

Last, we shall consider the existence of periodic solutions of (1). Suppose that the functions f(t, x), g(t, y), and W(t, r) are smooth enough to ensure the uniqueness of solutions. Let f(t, x) and W(t, r) be periodic in t, with period 1. Then the following statement is true.

THEOREM 11. *Let the function* $W(t, r)$ *be continuous, nondecreasing in* r, *and defined on* $I \times R^+$. *Let the function* $V^*(t, x, y)$ *of* (3) *satisfy condition* (4). *Suppose that* $V(t, x, y) = 0$ *if and only if* $x = y$, *and let the differential equation* (5) *possess a periodic solution with period* 1. *Then, if the system* (2) *possesses a bounded, nondecreasing solution in the sense of B. Viswanatham* [5], *the system* (1) *has a periodic solution of period* 1.

Proof. Let $y(t)$ be the bounded, monotonic solution of (2), with $y(t_0) = y_0$. Consider the solution $x(t)$ of (1) such that $x(t_0) = y_0$. Define $m(t) = V(t, x(t), y(t))$. Then $m(t_0) = 0$. Suppose that $r(t)$ is a periodic solution of (5). It is possible to choose t_0 and r_0 such that $r(t) - r_0 \geq 0$. For each small, positive ε, let $r(t, \varepsilon)$ be any solution of $r' = W(t, r) + \varepsilon$ such that $r(t_0, \varepsilon) = r_0$. Defining $P(t, \varepsilon) = r(t, \varepsilon) - r_0$ and noting that $W(t, r)$ is nondecreasing in r, we find that

$$P'(t, \varepsilon) \geq W(t, P(t, \varepsilon)) + \varepsilon .$$

By proceeding as in the proof of Lemma 1, we can easily obtain the inequality

$$m(t) \leq P(t, \varepsilon) \qquad (t \geq t_0) .$$

But

$$\lim_{\varepsilon \to 0} P(t, \varepsilon) = \lim_{\varepsilon \to 0} r(t, \varepsilon) - r_0 = r(t) - r_0 .$$

Hence,

$$m(t) = V(t, x(t), y(t)) \leq r(t) - r_0 \qquad (t \geq t_0) . \tag{26}$$

Denote the point $x(t_0)$ by P_0, and the point $x(t_0 + 1)$ on the solution $x(t)$ by P_1. Let T be the transformation that takes any point P_0 to the corresponding point P_1 obtained by the above process. Since the function $f(t, x)$ is assumed to be periodic in t, with period 1, any solution passing through any point fixed under the transformation defined above is clearly a periodic solution. Hence it is enough to prove the existence of a fixed point under the above transformation. Since $r(t)$ is periodic and since $r(t_0) = r_0$, $r(t_0 + n) - r_0 = 0$ $(n = 0, 1, 2, \cdots)$. It therefore follows from (26) that

$$V(t_0 + n, x(t_0 + n), y(t_0 + n)) = 0 \qquad (n = 0, 1, 2, \cdots) ,$$

which implies that $x(t_0 + n) = y(t_0 + n)$. Since $y(t)$ is bounded and monotonic, it is clear that the points $x(t_0 + n)$ form a bounded, monotonic, denumerably infinite set. If their upper bound is also included, the set becomes inductively ordered. Hence the application of Lemma 7 in [4] yields a fixed point, and the theorem is proved.

Theorem 11 generalizes our result in [4].

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