AN EXISTENCE THEOREM FOR PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

H. W. Knobloch

1. INTRODUCTION

We consider a system of ordinary differential equations

$$\dot{x} = f(x, t) \qquad (x = (x_1, \cdots, x_n),\ f = (f_1, \cdots, f_n)),$$

where the f_i are continuous and satisfy a (local) Lipschitz-condition for (x, t) in some region $\Omega \times I$. Here Ω is assumed to be an open region in Euclidean n-space R^n, and I is the unit interval $0 \leq t \leq 1$, notation which we shall keep throughout this paper. A solution $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))$ of the differential equations is called *periodic*, if it satisfies the boundary conditions $\xi_i(0) = \xi_i(1)$ $(i = 1, \cdots, n)$. In the theorem that is formulated below, we give conditions in terms of the functions f_i that guarantee the existence of periodic solutions in subregions of $\Omega \times I$. In order to make the nature of these conditions clearer we introduce them here in a more geometric way. Let us assign to each solution of the differential equation the curve in R^{n+1} with the parametric representation $(\xi_1(t), \cdots, \xi_n(t), t)$ and with the orientation given by increasing t. Let Z be a subregion of $\Omega \times I$, and let T be a sufficiently smooth hypersurface that belongs to the boundary of Z. We shall say that T is of uniform type with respect to Z if there are no two solution curves which intersect with T in such a way that one curve arrives from the interior and the other arrives from the exterior of Z. In the notation of Ważewski, this means T does not contain *points de sortie* (points of egress) as well as *points d'entrée* (points of ingress); see [3, p. 280], [1, p. 179].

We now consider a region $Z \subseteq \Omega \times I$ that is bounded by cylindrical and plane hypersurfaces S_0, S_1, T_i and T_i^* $(i = 1, \cdots, n)$. The hypersurface S_0 is

$$\{(x, t): t = 0,\ \alpha_i \leq x_i \leq \beta_i\},$$

and S_1 is

$$\{(x, t): t = 1,\ \alpha_i \leq x_i \leq \beta_i\}, \quad \text{where } \alpha_i = \alpha_i(0) = \alpha_i(1),\ \beta_i = \beta_i(0) = \beta_i(1).$$

The hypersurfaces T_i, T_i^* are defined by equations of the form $x_i = \alpha_i(t)$, $x_i = \beta_i(t)$, respectively, with $\alpha_i(t) \leq \beta_i(t)$ and $\alpha_i(0) = \alpha_i(1)$, $\beta_i(0) = \beta_i(1)$ $(i = 1, \cdots, n)$. Our theorem can be stated as follows: *If* $T_i \cup T_i^*$ *is of uniform type* $(i = 1, \cdots, n)$, *then there exists a periodic solution of the system* $\dot{x} = f(x, t)$ *inside* Z. It should be noted that if all T_i, T_i^* are of the same uniform type, then our statement is an immediate consequence of Brouwer's fixed point theorem. If, for example, there are no points of egress on $\bigcup_{i=1}^n T_i \cup T_i^*$, any solution that starts at some point $(x, 0) \in S_0$ cannot leave Z except at some point $(x, \bar{x}) \in S_1$. The mapping $x \to \bar{x}$ is

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then a continuous mapping of the box Q: $\alpha_i \leq x_i \leq \beta_i$ into itself, and hence it has a fixed point. In our theorem, however, we need not assume that all T_i, T_i^* are of the same type. For example, if $n = 2$, solution curves may leave the region Z through T_1, T_1^*, and solution curves may enter Z through T_2, T_2^*. In general, there will be no mapping of S_0 into S_1 or of S_1 into S_0. Nevertheless, there exists a periodic solution.

Brouwer's theorem is the basic tool in our proof, but this theorem will not appear in its usual form. We use a version which is due to Miranda [2] and which can be stated as follows.

THEOREM (Miranda). *Let* $\phi_i(u_1, \cdots, u_n)$ $(i = 1, \cdots, n)$ *be* n *functions which are defined and continuous in the box* Q: $\alpha_i \leq u_i \leq \beta_i$. *If each* ϕ_i $(i = 1, \cdots, n)$ *has constant sign on each of the faces* $u_i = \alpha_i$, $u_i = \beta_i$ *of* Q *and these signs are opposite, then the functions* $\phi_1, \cdots, \phi_n$ *have at least one common zero in* Q.

It is clear from the formulation of this theorem that we wish to reduce the problem of finding periodic solutions to the problem of finding common zeros of some functions ϕ_i. The most natural way to do this is to consider the solutions of the differential equations as functions of the initial values $u = (u_1, \cdots, u_n)$, that is, to consider the functions $\xi_i(t, u)$ which are characterized by the solution property and by the condition $\xi_i(0, u) = u_i$, and then to choose $u_i - \xi_i(1, u)$ to be the function $\phi_i(u)$. But $\phi_i(u)$ may not exist for all $u \in Q$, and even if it does, the conditions of Miranda's theorem are in general not satisfied. The idea of our proof is to replace each right member f_i of the differential equations by a function $\tilde{f}_i$ that differs from f_i only outside Z, but for which the corresponding periodic solutions (if there are any) of the new system $\dot{x} = \tilde{f}(x, t)$ lie inside Z. Then we treat this new system $\dot{x} = \tilde{f}(x, t)$ as indicated above.

2. THE THEOREM

We first introduce some notation. Let $\pi_i(x, t)$ denote the n-tuple

$$(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t),$$

and let $\pi_i(X)$ denote the set of all $\pi_i(x, t)$ with $(x, t) \in X$, where $X \subseteq R^n \times I$.

A function ϕ, defined on some set X, is said to have constant sign on X if either $\phi(x) \geq 0$ or $\phi(x) \leq 0$ for all $x \in X$. If two functions ϕ, ψ, defined on sets X, Y, respectively, both have constant sign and one of the two combinations $\{\phi \geq 0, \psi \leq 0\}$, $\{\phi \leq 0, \psi \geq 0\}$ holds, then we shall say that ϕ and ψ have opposite constant signs.

THEOREM. *Let* $f(x, t) = (f_1(x, t), \cdots, f_n(x, t))$, *where the* f_i *are defined, continuous, and satisfy a local Lipschitz condition with respect to* x *on some set* $\Omega \times I$ (Ω *an open set in* R^n). *For each* $i = 1, \cdots, n$ *suppose there exists a pair of functions* $\alpha_i(t)$ *and* $\beta_i(t)$, *with* $\alpha_i(t) \leq \beta_i(t)$, *that are defined on* I, *continuous, periodic with period* 1, *and piecewise continuously differentiable. Let* Z *be the subregion of* $R^n \times I$ *that is defined by the inequalities*

$$\alpha_i(t) \leq x_i \leq \beta_i(t) \qquad (i = 1, \cdots, n),$$

and let D *be a finite subset of* I *such that* $\alpha_i(t)$, $\beta_i(t)$ *are differentiable outside* D $(i = 1, \cdots, n)$. *Lastly, we assume that*

(a) $Z \subseteq \Omega \times I$,

(b) *The two functions* (*of the* n *variables* $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t$)

$$\dot{\alpha}_i(t) - f_i(x_1, \cdots, x_{i-1}, \alpha_i(t), x_{i+1}, \cdots, x_n, t)$$

$$\dot{\beta}_i(t) - f_i(x_1, \cdots, x_{i-1}, \beta_i(t), x_{i+1}, \cdots, x_n, t)$$

have opposite constant signs on $\pi_i(Z \cap R^n \times (I - D))$ $(i = 1, \cdots, n)$.

Then the differential system $\dot{x} = f(x, t)$ *has at least one periodic solution in* Z.

For the proof of this theorem we require some lemmas.

3. LEMMAS

Let $X_i(u, t)$ be the following function of the two variables u, t:

$$X_i(u, t) = \begin{cases} \alpha_i(t), & \text{if } u < \alpha_i(t), \\ u, & \text{if } \alpha_i(t) \leq u \leq \beta_i(t), \\ \beta_i(t), & \text{if } \beta_i(t) < u. \end{cases}$$

This function is defined for all u and all $t \in I$, is continuous, and satisfies the Lipschitz condition.

$$|X_i(u, t) - X_i(v, t)| \leq |u - v| \tag{1}$$

for all u, v.

The $n + 1$-tuple $(X_1(x_1, t), \cdots, X_n(x_n, t), t)$ is always a point in Z; hence, the functions

$$\tilde{f}_i(x_1, \cdots, x_n, t) = f_i(X_1(x_1, t), \cdots, X_n(x_n, t), t) \qquad (i = 1, \cdots, n) \tag{2}$$

are defined and continuous in $R^n \times I$. We denote the vector function $(\tilde{f}_1, \cdots, \tilde{f}_n)$ by $\tilde{f}$.

Our proof is based essentially on the following four properties of f(x, t).

LEMMA 1. (i) *If* $(x, t) \in Z$, $\tilde{f}(x, t) = f(x, t)$.

(ii) *The function* $\dot{\alpha}_i(t) - \tilde{f}_i(x, t)$ *has constant sign on the set of all* (x, t) *with* $x_i \leq \alpha_i(t)$, $t \notin D$; *the function* $\dot{\beta}_i(t) - \tilde{f}_i(x, t)$ *has the opposite sign on the set of all* (x, t) *with* $x_i \geq \beta_i(t)$, $t \notin D$.

(iii) *The functions* $|f_i|$ *are bounded on* $R^n \times I$ $(i = 1, \cdots, n)$, *and*

(iv) f_i *satisfies a Lipschitz condition everywhere.*

Proof. The statement (i) follows from the fact that

$$(x, t) = (X_1(x_1, t), \cdots, X_n(x_n, t)) \qquad ((x, t) \in Z).$$

Conclusion (ii) is true because of hypothesis (b) of the Theorem. For if $x_i \leq \alpha_i(t)$ $(x_i \geq \beta_i(t))$,

$$X_i(x_i, t) = \alpha_i(t) \quad (X_i(x_i, t) = \beta_i(t)).$$

Thus

$$\dot{\alpha}_i(t) - \tilde{f}_i(x, t) = \dot{\alpha}_i(t) - f_i(X_1, \cdots, X_{i-1}, \alpha_i(t), X_{i+1}, \cdots, X_n, t) \qquad (x_i \leq \alpha_i(t)),$$

and

$$\dot{\beta}_i - \tilde{f}_i(x, t) = \dot{\beta}_i(t) - f_i(X_1, \cdots, X_{i-1}, \beta_i(t), X_{i+1}, \cdots, X_n, t) \qquad (x_i \geq \beta_i(t)),$$

$(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n, t)$ being a point in $\pi_i(Z)$. That $|f_i|$ is bounded on $R^n \times I$ is evident since $(X_1(x_1, t), \cdots, X_n(x_n, t), t)$ is always in Z, and Z is bounded.

The last statement is a consequence of (1). By our assumptions on f(x, t) and since Z is bounded, there is a constant L such that

$$|f_i(x, t) - f_i(y, t)| \leq L \max_j |x_j - y_j|$$

for all (x, t), (y, t) $\in$ Z.

From this we see, in view of (1) and (2), that

$$|\tilde{f}_i(x, t) - \tilde{f}_i(y, t)| \leq L \max |X_j(x_j, t) - X_j(y_j, t)|$$
$$\leq \max |x_j - y_j|.$$

LEMMA 2. *The differential system*

$$\dot{x} = \tilde{f}(x, t) \tag{3}$$

has a periodic solution in Z *if it has any periodic solution.*

Proof. Let $\xi = (\xi_1, \cdots, \xi_n)$ be a periodic solution of (3). Each function

$$\delta_i = \xi_i - \alpha_i$$

satisfies the condition $\delta_i(0) = \delta_i(1)$, and hence it can be extended to a periodic function of period 1, defined, continuous, and piecewise continuously differentiable for all t. For $0 \leq t \leq 1$,

$$\dot{\delta}_i(t) = \dot{\xi}_i(t) - \dot{\alpha}_i(t) = \tilde{f}_i(\xi_1(t), \cdots, \xi_n(t), t) - \dot{\alpha}_i(t).$$

In view of conclusion (ii) of Lemma 1, the last expression has constant sign on the set of all $t \notin D$, for which $\xi_i(t) \leq \alpha_i(t)$, and therefore $\dot{\delta}_i(t)$ has constant sign on the set of all $t \notin D$ for which $\delta_i(t) \leq 0$. Consequently, δ_i itself must have constant sign for all t. Otherwise, since δ_i is periodic, for each t_0 the function δ_i would change sign on the half-line $t \geq t_0$ as well as on the half-line $t \leq t_0$. Now, if t_0 is a point at which δ_i is negative, then there exists an interval $[t_1, t_2]$ around this point such that $\delta_i(t_1) = \delta_i(t_2) = 0$ and $\delta_i(t) < 0$ for $t_1 < t < t_2$. The inequality $\delta_i(t) < 0$ implies, as we have seen before, that $\dot{\delta}_i$ has constant sign, and hence δ_i is monotonic in this interval. This is obviously not compatible with the conditions that $\delta_i(t_1) = \delta_i(t_2) = 0$ and $\delta_i(t) < 0$.

Consequently there are two possibilities:

$\delta_i(t) \geq 0$ for all t, that is, $\xi_i(t) \geq \alpha_i(t)$, or

$\delta_i(t) \leq 0$ for all t.

In the second case, δ_i would be a monotonic periodic function of t (since $\dot{\delta}_i(t)$ would have constant sign for all $t \notin D$). Hence it would be a nonpositive constant c, that is,

$$\xi_i(t) = \alpha_i(t) - c \qquad (c \geq 0).$$

Since

$$\frac{d}{dt}(\alpha_i(t) - c) = \dot{\alpha}_i(t), \text{ and}$$

$$X_i(\alpha_i(t) - c, t) = \alpha_i(t),$$

the relations

$$\dot{\xi}_j = \tilde{f}_j(\xi_1, \cdots, \xi_n, t) \qquad (j = 1, \cdots, n)$$

remain unchanged if ξ_i is replaced by α_i. This means that

$$(\xi_1, \cdots, \xi_{i-1}, \alpha_i, \xi_{i+1}, \cdots, \xi_n),$$

is also a periodic solution of (3). If we apply similar arguments to the difference $\xi_i - \beta_i$, we arrive at the conclusion that either

$$\alpha_i(t) \leq \xi_i(t) \leq \beta_i(t)$$

for all t or $\xi_i(t)$ can be replaced by $\alpha_i(t)$ or $\beta_i(t)$ and this gives another periodic solution of (3). Thus the lemma is proved.

4. PROOF OF THE THEOREM

In view of part (i) of Lemma 1, each periodic solution of (3) that lies in Z is, in fact, a periodic solution of the original system $\dot{x} = f(x, t)$. In order to complete the proof of our theorem, we shall now show that there exists at least one periodic solution of the differential system (3).

We observe from parts (iii) and (iv) of Lemma 1 that a solution of the differential equation (3) can always be extended to the whole interval I. There exists, therefore, a vector function

$$\tilde{\xi}(t, u) = (\tilde{\xi}_1(t, u), \cdots, \tilde{\xi}_n(t, u)),$$

with components $\tilde{\xi}_i(t, u)$, that are defined and continuous for all $t \in I$ and all $u = (u_1, \cdots, u_n)$, such that

$$\frac{d}{dt}\tilde{\xi}(t, u) = \tilde{f}(\tilde{\xi}(t, u), t),$$

$$\tilde{\xi}(0, u) = u.$$

The n functions ϕ_i defined by the equations

$$\phi_i(u) = u_i - \tilde{\xi}_i(1, u) \qquad (i = 1, \cdots, n)$$

are then defined and continuous everywhere.

Now we are ready to proceed in the straightforward way outlined in the introduction. For brevity, let us write α_{i0} for $\alpha_i(0)$ $(= \alpha_i(1))$ and β_{i0} for $\beta_i(0)$ $(= \beta_i(1))$. We will show that each ϕ_i has opposite constant sign on the hyperplanes $u_i = \alpha_{i0}$ and $u_i = \beta_{i0}$. It will then follow from Miranda's theorem that the system of equations $\phi_i(u) = 0$ has a solution and that this solution represents the initial values of a periodic solution of the differential system (3).

Let us consider the functions

$$\eta_i(t, u) = \alpha_i(t) - \tilde{\xi}_i(t, u),$$

$$\eta_i^*(t, u) = \beta_i(t) - \tilde{\xi}_i(t, u).$$

First of all,

$$(4) \qquad \left.\begin{aligned} \eta_i(0, u) &= 0 \\ \eta_i(1, u) &= \phi_i(u) \end{aligned}\right\} \text{ for } u_i = \alpha_{i0}; \qquad \left.\begin{aligned} \eta_i^*(0, u) &= 0 \\ \eta_i^*(1, u) &= \phi_i(u) \end{aligned}\right\} \text{ for } u_i = \beta_{i0}.$$

Furthermore, we shall show that

(a) η_i has constant sign on $\{(t, u): u_i = \alpha_{i0}\}$ and this sign is the sign of $\dot{\alpha}_i - \tilde{f}_i$ on $\{(x, t): x_i \leq \alpha_i(t)\}$, and

(b) η_i^* has constant sign on $\{(t, u): u_i = \beta_{i0}\}$ and this sign is the sign of $\dot{\beta}_i - \tilde{f}_i$ on $\{(x, t): x_i \geq \beta_i(t)\}$.

Once we have proved this, by part (ii) of Lemma 1, it follows that η_i and η_i^* have opposite signs, and hence, in view of (4), we obtain the desired property of ϕ_i.

We present here only the proof of statement (a) since the procedure for proving (b) is analogous.

In what follows we regard u as fixed and assume $u_i = \alpha_{i0}$. We write $\eta(t)$ for $\eta_i(t, u)$ and $F(w, t)$ for

$$\dot{\alpha}_i(t) - \tilde{f}_i(\tilde{\xi}_1(t, u), \cdots, \tilde{\xi}_{i-1}(t, u), -w + \alpha_i(t), \tilde{\xi}_{i+1}(t, u), \cdots, \tilde{\xi}_n(t, u), t).$$

The function η satisfies the differential equation $\dot{w} = F(w, t)$ provided $t \notin D$. Furthermore, in view of (ii), $F(w, t)$ has constant sign on $\{(w, t): w \geq 0\}$, and this sign is the same as the sign of $\dot{\alpha}_i(t) - \tilde{f}_i(x, t)$ on $\{(x, t): x_i \leq \alpha_i(t), \ t \notin D\}$.

Case 1. $F(w, t) \leq 0$ for $w \geq 0$. In this case, $\eta(t)$ must be nonpositive for all $t \in I$. Otherwise, since $\eta(0) = 0$, there would be a $t \in I$, $(t \notin D)$ such that $\eta(t) > 0$, $\dot{\eta}(t) > 0$, and this would not be compatible with the relation $\dot{\eta} = F(\eta, t)$.

Case 2. $F(w, t) \geq 0$ for $w \geq 0$. Let $[t_0, t_1]$ be a subinterval of I, that contains no point of D in its interior. The function $F(w, t)$ is then continuous and bounded for all $t \in [t_0, t_1]$ and all w. Furthermore, F satisfies a Lipschitz condition with respect to w. Therefore, according to the Picard-Lindelöf Theorem,

$$\eta(t) = \lim_{\nu \to \infty} \eta_\nu(t),$$

where the sequence $\eta_\nu(t)$ is defined by the formulas

$$\eta_0(t) = \eta(t_0),$$

$$\eta_{\nu+1}(t) = \eta(t_0) + \int_{t_0}^{t} F(\eta_\nu(\tau), \tau)\, d\tau .$$

It follows immediately from this and our assumption on F, that if $\eta(t_0) \geq 0$, $\eta_\nu(t) \geq 0$, and hence $\eta(t) \geq 0$ for all $t \in [t_0, t_1]$. We thus arrive at the conclusion that $\eta(t_0) \geq 0$ implies that $\eta(t) \geq 0$ for all $t \in [t_0, t_1]$. Since $\eta(0) = 0$, $\eta(t) \geq 0$ in the interval between 0 and the nearest point of D. In particular, $\eta \geq 0$ at this point, and so $\eta(t) \geq 0$ in the interval starting from there and going to the next point of D. Hence, finally, $\eta(t) \geq 0$ for all $t \in I$.

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The University of Michigan
and
Universität München