ON AN INVARIANT PROPERTY OF SURFACE INTEGRALS

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Our basic tool is the following proposition.

LEMMA. *If* $\alpha = (a_{ij})$ *is an* $n \times n$ *orthogonal matrix, and* $\beta = (b_{\xi\eta})$ *denotes the* $\binom{n}{2} \times \binom{n}{2}$ *matrix whose elements* $b_{\xi\eta}$ *are the determinants of all* 2×2 *submatrices of* α, *then* β *is also an orthogonal matrix.*

We give a proof in Section 2. In Section 3, we use this result to extend the theorems of L. H. Turner [7] concerning the invariance of Cesari's surface integral under orthogonal linear transformations.

1. NOTATION

Let E_n $(n \geq 2)$ be the n-dimensional Euclidean space with an orientation.

If n is a positive integer $(n \geq 2)$, then Ω_n^2 denotes the set of all ordered pairs $\xi = (\xi^1, \xi^2)$ of integers such that $1 \leq \xi^1 < \xi^2 \leq n$. We shall assume that Ω_n^2 is lexiographically ordered.

By the mapping P_n^ξ $(\xi \in \Omega_n^2)$ we mean the projection

$$P_n^\xi(x) = (x^{\xi^1}, x^{\xi^2}) \qquad (x = (x^1, x^2, \cdots, x^n) \in E_n)$$

of E_n onto the hyperplane E_2^ξ.

Let (T, A): $x = T(w)$ $(w \in A)$ be any continuous mapping from an admissible set $A \subset E_2$ into E_n $(n \geq 2)$. Denote by (T^ξ, A) $(\xi \in \Omega_n^2)$ the $\binom{n}{2}$ plane mappings $(P_n^\xi T, A)$ from the admissible set $A \subset E_2$ into $E_2^\xi \subset E_n$. Let $\mathfrak{S}$ be any set of non-overlapping closed simple polygonal regions π in A. If π^* is the oriented boundary of π, then T^ξ maps π^* into an oriented closed curve C_π^ξ in E_2^ξ. For any point $x \in E_2^\xi$, let $O(x; C_\pi^\xi)$ be the topological index of x with respect to C_π^ξ. Then $O(x; C_\pi^\xi)$ is Borel measurable and integrable if (T, A) is cBV. We write

$$u(T^\xi, \pi) = (E_2^\xi)\int O(x; C_\pi^\xi) \quad \text{and} \quad u(T, \pi) = \left[\sum u^2(T^\xi, \pi)\right]^{1/2},$$

where Σ ranges over $\xi \in \Omega_n^2$. (See [1] for the definitions of admissible sets, topological index, and cBV.)

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2. PROOF OF THE LEMMA

β is the $\binom{n}{2} \times \binom{n}{2}$ matrix $\beta = (b_{\xi\eta})$, where

$$b_{\xi\eta} = \begin{vmatrix} a_{ik} & a_{im} \\ a_{jk} & a_{jm} \end{vmatrix} \qquad (\xi = (i, j),\ \eta = (k, m) \in \Omega_n^2).$$

Since α is orthogonal,

$$\sum_{j=1}^{n} a_{ij}^2 = 1 \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} a_{kj} = 0 \quad (i \neq k).$$

We show that the row vectors of β form an orthogonal system of $1 \times \binom{n}{2}$ vectors. Clearly,

$$1 = \left(\sum_{k=1}^{n} a_{ik}^2\right)\left(\sum_{m=1}^{n} a_{jm}^2\right) - \left(\sum_{m=1}^{n} a_{im} a_{jm}\right)^2$$

$$= \sum_{k<m} (a_{ik}^2 a_{jm}^2 + a_{im}^2 a_{jk}^2) - 2 \sum_{k<m} (a_{ik} a_{jk} a_{jm} a_{im})$$

$$= \sum_{k<m} (a_{ik}^2 a_{jm}^2 - 2a_{ik} a_{jk} a_{im} a_{jm} + a_{im}^2 a_{jk}^2) = \sum_{\eta} b_{\xi\eta}^2 ,$$

where $\xi = (i, j)$ and $\eta = (k, m)$. Hence the row vectors are normal. Since

$$0 = \left(\sum_{k=1}^{n} a_{ik} a_{sk}\right)\left(\sum_{m=1}^{n} a_{jm} a_{tm}\right) - \left(\sum_{k=1}^{n} a_{ik} a_{tk}\right)\left(\sum_{m=1}^{n} a_{jm} a_{sm}\right)$$

$$= \sum_{k<m} (a_{ik} a_{sk} a_{jm} a_{tm} + a_{im} a_{sm} a_{jk} a_{tk}) - \sum_{k<m} (a_{ik} a_{tk} a_{jm} a_{sm} + a_{im} a_{tm} a_{jk} a_{sk})$$

$$= \sum_{\eta} (a_{ik} a_{jm} - a_{jk} a_{im})(a_{sk} a_{tm} - a_{tk} a_{sm}) = \sum_{\eta} b_{\xi\eta} b_{\zeta\eta} ,$$

where $\xi = (i, j)$, $\zeta = (s, t)$, $\eta = (k, m)$, and $\xi \neq \zeta$, the row vectors are orthogonal. This concludes the proof of the lemma.

Let π^* be the oriented boundary of a closed simple polygonal region π in the plane E_2, and let f be a continuous mapping of π^* into E_n such that the oriented curve (f, π^*) is rectifiable. If $\xi \in \Omega_n^2$, then $(P_n^\xi f, \pi^*)$ is also rectifiable, and by [1, (8.10.i)] the integral

$$u(P_n^\xi f) = u(P_n^\xi f, \pi) = (E_2^\xi) \int O(x;\ P_n^\xi f, \pi^*)$$

exists. Hence, corresponding to f, there exists an $\binom{n}{2} \times 1$ column vector

$$Z(f) = \mathrm{col}\,(u(P_n^{\xi_1} f),\ u(P_n^{\xi_2} f),\ \cdots,\ u(P_n^{\xi_N} f)) \qquad \left(\xi_1 < \xi_2 < \cdots < \xi_N \in \Omega_n^2,\ N = \binom{n}{2}\right)$$

whose Euclidean norm is $u(f) = u(f, \pi)$. The next theorem generalizes a theorem of L. Cesari [1 (8.11.i)].

THEOREM 1. *Let* (f, π^*) *be as above, let* $\alpha = (a_{ij})$ *be an orthogonal linear transformation of* E_n *onto itself, and let* $\beta = (b_{\xi\eta})$ *be the* $\binom{n}{2} \times \binom{n}{2}$ *matrix defined in the lemma. Then* $(\alpha f, \pi^*)$ *is rectifiable,* $Z(\alpha f) = \beta Z(f)$, *and* $u(f) = u(\alpha f)$.

Proof. Clearly, $(\alpha f, \pi^*)$ is rectifiable. Also, since β is orthogonal, $u(f) = u(\alpha f)$ if $Z(\alpha f) = \beta Z(f)$. Hence, we need only prove that $Z(\alpha f) = \beta Z(f)$.

Now f is a vector function; namely, $f = \mathrm{col}\,(f^1, f^2, \cdots, f^n)$. Let us denote αf by the vector function $\alpha f = \mathrm{col}\,(F^1, F^2, \cdots, F^n)$, where

$$F^i = \sum_{j=1}^{n} a_{ij} f^j \qquad (i = 1, 2, \cdots, n).$$

By [1, (8.10.i)], $2\,u(P_n^{\xi} \alpha f) = \int F^i\, dF^j$, where the integral is taken over π^* and $\xi = (i, j) \in \Omega_n^2$. Expanding the integral, we see that

$$\begin{aligned} \int F^i\, dF^j &= \int \left(\sum_{k=1}^{n} a_{ik} f^k\right)\left(\sum_{m=1}^{n} a_{jm}\, df^m\right) \\ &= \sum_{m=1}^{n} a_{im} a_{jm} \int f^m\, df^m + \sum_{k<m} (a_{ik} a_{jm} - a_{im} a_{jk}) \int f^k\, df^m \\ &= 2 \sum_{\eta} b_{\xi\eta}\, u(P_n^{\eta} f), \end{aligned}$$

where $\xi = (i, j)$, $\eta = (k, m) \in \Omega_n^2$. Hence $Z(\alpha f) = \beta Z(f)$, and Theorem 1 is proved.

3. THE INVARIANCE OF THE CESARI SURFACE INTEGRAL

If (T, A) is a cBV mapping, then each of the plane mappings (T^{ξ}, A) $(\xi \in \Omega_n^2)$ has finite variation (or area) $V(T^{\xi}, A)$. Each of these variations is equal to the sum of a positive variation and a negative variation: $V(T^{\xi}, A) = V^+(T^{\xi}, A) + V^-(T^{\xi}, A)$. The relative variation is defined as $\mathscr{V}(T^{\xi}, A) = V^+(T^{\xi}, A) - V^-(T^{\xi}, A)$. Let $\mathscr{V}(T, A)$ be the column vector $\mathrm{col}\,(\mathscr{V}(T^{\xi_1}, A),\ \mathscr{V}(T^{\xi_2}, A),\ \cdots,\ \mathscr{V}(T^{\xi_N}, A))$, where $\xi_1 < \xi_2 < \cdots < \xi_N \in \Omega_n^2$, $N = \binom{n}{2}$.

The following theorem is an application of Theorem 1. When n = 3, Theorem 2 reduces to [7, Theorem 3].

THEOREM 2. *Let* (T, A) *be a* cBV *mapping, and let* α *be an orthogonal linear transformation of* E_n *onto itself. Then* $\mathcal{V}(\alpha T, A) = \beta \mathcal{V}(T, A)$.

The proof of Theorem 2 is essentially the same as that given in [7, Section 3]. Two changes must be made. The reference to [1, page 104] is replaced by Theorem 1 above. The second change is the reference to the equality of the Lebesgue area L(T, A) and V(T, A). In this case, references are made to [2], [3, Theorem 7.14], [4], and [5].

Let $\mathfrak{S}$ be a finite system of nonoverlapping closed simple polygonal regions $\pi \subset A$, and let Σ_π denote a sum over $\pi \in \mathfrak{S}$. For the system $\mathfrak{S}$ and the cBV mapping (T, A), we define three nonnegative indices d, m, μ as follows:

$d = \max[\operatorname{diam} T(\pi): \pi \in \mathfrak{S}]$;

$m = \max[|T^{\xi}(\bigcup \pi^*)|: \xi \in \Omega_n^2]$, where the absolute value sign denotes two-dimensional Lebesgue measure and $\bigcup$ ranges over $\pi \in \mathfrak{S}$;

$\mu = \max[V(T, A) - \Sigma_\pi u(T, \pi), V(T^{\xi}, A) - \Sigma_\pi |u(T^{\xi}, \pi)| \quad (\xi \in \Omega_n^2)]$.

From [6, Theorem 3.i] we see that, for each cBV mapping (T, A), there exist systems $\mathfrak{S}$ with arbitrarily small indices d, m, μ.

Let f(x, d) be a continuous function of (x, d), where x ranges over some set $K \subset E_n$ and d is any point of E_N $\left(N = \binom{n}{2}\right)$. We call f(x, d) a *parametric integrand* if f(x, d) is positively homogeneous in d; that is, if f(x, td) = tf(x, d) for all $x \in K$, $t \geq 0$, and $d \in E_N$. By $\|d\|$ we shall denote the Euclidean norm of d. Let d_π be the vector $(u(T^{\xi_1}, \pi), u(T^{\xi_2}, \pi), \cdots, u(T^{\xi_N}, \pi))$, where $\xi_1 < \xi_2 < \cdots < \xi_N \in \Omega_n^2$ and $N = \binom{n}{2}$. Then the following existence theorem for the Cesari surface integral is proved in the same manner as in [1, Appendix B].

THEOREM 3. *Let* (T, A) *be a* cBV *mapping. Let* f(x, d) *be a parametric integrand defined on* $K \subset E_N$ $\left(N = \binom{n}{2}\right)$ *such that* $T(A) \subset K$ *and* f(x, d) *is bounded and uniformly continuous on* $R = \{(x, d): x \in T(A), \|d\| = 1\}$. *Then the limit* $I(T, A, f) = \lim \Sigma_\pi f(x_\pi, d_\pi)$ *exists, where* x_π *is any point of* $T(\pi)$, π *is an element of* $\mathfrak{S}$, *and the limit is taken as the indices* d, m, μ *of* $\mathfrak{S}$ *tend to zero.*

The invariance of the Cesari surface integral can now be established in exactly the same manner as in [7, Theorem 4]. Theorem 2 replaces [7, Theorem 3] in Turner's proof.

THEOREM 4. *Let* (T, A) *and* f(x, d) *be as above. Let* α *be an orthogonal linear transformation of* E_n *onto itself, and let* β *be the orthogonal linear transformation of* E_N *onto itself given in the lemma above. Let* $g(x, d) = f(\alpha^{-1}x, \beta^{-1}d)$ *on* $(\alpha K) \times E_N$. *Then* $I(\alpha T, A, g)$ *exists and equals* I(T, A, f).

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