THE TAYLOR COEFFICIENTS OF INNER FUNCTIONS

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INTRODUCTION

The object of the present paper is to study bounds on the Taylor coefficients of a function f(z) that is regular and bounded by 1 in $|z| < 1$ and has boundary values $f(e^{i\theta})$ of modulus 1 for almost all θ. We call such a function an *inner function* (terminology introduced by Beurling [1]). Inner functions play an important role in the study of functions of class H_p (see for example Privalov [6, p. 53], Zygmund [9, Vol. I, p. 271]), in certain approximation questions [1], and in the study of the invariant subspaces of the "shift operator" in ℓ_2 [1]. It is known [6] that the most general inner function is the product of a Blaschke product and a function of the form

$$\exp\left(\int_o^{2\pi} \frac{z + e^{it}}{z - e^{it}} d\rho(t)\right),$$

where $\rho(t)$ is a positive measure singular with respect to Lebesgue measure. The set of Taylor coefficients of an inner function can also be described, without reference to analytic functions, as a solution of the infinite system of quadratic equations

$$\sum_{n=0}^{\infty} |a_n|^2 = 1,$$

$$\sum_{n=0}^{\infty} a_n \bar{a}_{n+k} = 0 \qquad (k = 1, 2, \cdots).$$

Qualitatively, our main results are these: the coefficients of an inner function that is not a finite Blaschke product cannot be $o(1/n)$, although they can be $O(1/n)$; and if the function does not vanish in $|z| < 1$, they are sometimes $O(n^{-3/4})$ and never $o(n^{-3/4})$.

1. COEFFICIENTS OF INNER FUNCTIONS

THEOREM 1. *Let* $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ *be an inner function, and denote by* A_n *the infinite matrix*

$$\begin{pmatrix} |a_n| & |a_{n+1}| & |a_{n+2}| & \cdots \\ |a_{n+1}| & |a_{n+2}| & |a_{n+3}| & \cdots \\ |a_{n+2}| & |a_{n+3}| & |a_{n+4}| & \cdots \\ \cdots & & & \end{pmatrix}.$$

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If for some $n > 0$ *the quadratic form with matrix* A_n *is bounded, and its bound is less than* 1, *then* $f(z)$ *is a rational function (finite Blaschke product).* (By a somewhat more delicate argument, which we shall not give here, the conclusion may be shown to hold even when the bound equals 1, under the assumption that f is a Blaschke product.)

Before proceeding to the proof, we note two cases in which the hypothesis concerning A_n is satisfied.

(i) Let $|a_n| \leq C(n+1)^{-1} + b_n$, where $C < 1/\pi$ and $b_n \geq 0$, $\Sigma b_n < \infty$. Indeed, in in this case the form with matrix A_n is majorized by the form with matrix $CH_n + B_n$, where $H_n = \|(n+i+j-1)^{-1}\|$ $(i, j = 1, 2, \cdots)$ is a section of the Hilbert matrix and has bound at most π (it is easy to show that the bound is equal to π) and where $B_n = \|b_{n+i+j-2}\|$. By a theorem of Schur [2, p. 198], B_n has a bound not exceeding its largest row sum; for large n this is arbitrarily small, and in particular less than $1 - C\pi$. For such n, A_n has norm less than 1. We see in particular that an inner function for which $a_n = o(1/n)$ (and even an inner function for which $\limsup n|a_n| < 1/\pi$) must be a finite Blaschke product.

(ii) Let $\Sigma_1^\infty n|a_n|^2 < \infty$. Here the sum of the squares of all the entries of the matrix A_n is arbitrarily small for large n. As is well known, the square root of the sum (Hilbert-Schmidt norm) is an upper found for the form with matrix A_n. We thus see that an inner function with finite Dirichlet integral is a finite Blaschke product; that is, a nonrational inner function maps $|z| < 1$ onto a Riemann surface of infinite area.

Proof of Theorem 1. Suppose $f(z)$ is not a finite Blaschke product. Then $f(z_n) \to 0$ for some sequence $\{z_n\}$ $(|z_n| \to 1)$. For, in the contrary case, $|f(z)| \geq \delta > 0$ when $|z| > r_0$. This implies $f(z)$ has at most finitely many zeros in $|z| < 1$. Let $B(z)$ denote the Blaschke product with these zeros. Then $F(z) = B(z)/f(z)$ is regular and bounded in $|z| < 1$, and $|F(e^{i\theta})| = 1$ a.e. Hence $|F(z)| \leq 1$ in $|z| < 1$. In like manner $F(z)^{-1}$ is regular and bounded by 1 in $|z| < 1$. Hence F is a constant, a contradiction.

Let now $g(z)$ be any function of class H_∞ (bounded analytic functions, with the sup norm). Then $\|z^n - f(z)g(z)\|_\infty = 1$. Thus, the distance (based on the norm of ess sup on $|z| = 1$) from $e^{in\theta}\bar{f}(e^{i\theta})$ to the set of boundary functions of class H_∞ is 1. By the duality theorems of Havinson [3] and of Rogosinski and Shapiro [7], there exists for every $\varepsilon > 0$ a function $h(z) = h_\varepsilon(z)$ of class H_1 satisfying

$$(1) \qquad \|h\|_1 = 1, \qquad \left| \frac{1}{2\pi} \int_0^{2\pi} e^{i(n+1)\theta} \bar{f}(e^{i\theta})\, h(e^{i\theta})\, d\theta \right| > 1 - \varepsilon .$$

Now (see Zygmund [9, Vol. I, p. 275]) we may write

$$h(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \sum_{k=0}^{n} p_k q_{n-k}, \qquad \sum_{n=0}^{\infty} |p_n|^2 = \sum_{n=0}^{\infty} |q_n|^2 = 1 .$$

Further, the left side of (1) is formally equal to

$$|b_o \bar{a}_{n+1} + b_1 \bar{a}_{n+2} + \cdots| = \left| \sum_{k=0}^{\infty} \bar{a}_{n+k+1} \sum_{i+j=k} p_i q_j \right| .$$

The last member is a bilinear expression in the p_i and q_j, and the absolute values of the coefficients are precisely the elements of the matrix A_{n+1}. Hence the last series is absolutely convergent; this justifies the formal manipulation, and the sum cannot exceed the bound of A_{n+1}. Hence this bound exceeds $1 - \varepsilon$. Since ε is arbitrary, the bound is at least 1. This is a contradiction, and the theorem is proved.

THEOREM 2. *There exists an infinite Blaschke product with Taylor coefficients* $O(1/n)$.

LEMMA 1. *Let* $u(t)$ *be nonnegative and of class* C^1 *on* $0 \leq t < \infty$, *and let* $\lim_{t\to\infty} t\,u(t) = \lim_{t\to\infty} t\,u'(t) = 0$. *Then*

$$\sum_{k=1}^{\infty} u(k) \leq V(u) + \int_o^{\infty} u(t)\,dt ,$$

where V *denotes total variation.*

Proof. Clearly,

$$\sum_{k=1}^{\infty} u(k) = \int_o^{\infty} u(t)\,d[t] = - \int_o^{\infty} [t]\,u'(t)\,dt$$

$$= \int_o^{\infty} (t - [t])\,u'(t)\,dt - \int_o^{\infty} t\,u'(t)\,dt .$$

The first of the integrals in the last member is bounded by $V(u)$, the second equals $\int_o^{\infty} u(t)\,dt$, and the lemma is proved.

LEMMA 2. *Let* $f(z)$ *denote a Blaschke product whose zeros* z_n *satisfy*

$$1 - |z_{n+1}| \leq a(1 - |z_n|)$$

for some a $(0 < a < 1)$. *Then* $|f'(z_n)| > \dfrac{b}{1 - |z_n|}$, *where* b *is a positive constant.*

We refer to Newman [5] for the simple proof.

Proof of Theorem 2. Let $f(z)$ be the Blaschke product with zeros at $z_k = 1 - e^{-k}$ $(k = 1, 2, \cdots)$. Then

$$a_n = \frac{1}{2\pi} \int_o^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta .$$

Hence

$$\bar{a}_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{z^{n-1}}{f(z)}\, dz,$$

and simple considerations concerning the boundary behavior of f(z) justify computation of this integral by residues. We get

$$\bar{a}_n = \sum_{k=1}^{\infty} \frac{z_k^{n-1}}{f'(z_k)},$$

whence, by Lemma 2,

$$|a_{n+1}| \leq \frac{1}{b}\sum_{k=1}^{\infty} (1 - z_k)\, z_k^n,$$

and it suffices to show that $\Sigma_{k=1}^{\infty} u(k) < C/n$, where $u(t) = e^{-t}(1 - e^{-t})^n$. Now

$$V(u) = 2 \max u = \frac{2}{n+1}\left(1 - \frac{1}{n+1}\right)^n = O(1/n),$$

and

$$\int_0^\infty u(t)\, dt = \frac{1}{n+1}.$$

The theorem is proved.

Remark. It is clear from the construction that $a_n = O(1/n)$ for any Blaschke product whose zeros satisfy the hypothesis of Lemma 2. We have not been able to determine the lower bound of numbers C such that there exists an infinite Blaschke product with $\limsup n\,|a_n| = C$.

THEOREM 3. *Let* $f(z) = \Sigma_0^\infty a_n z^n$ *be an inner function for which* $z = 1$ *is a regular point. If, for some* k, $\Delta^k a_n = o(1/n)$, *then* f *is a finite Blaschke product (here* Δ *denotes the difference operator:* $\Delta a_n = a_n - a_{n+1}$*).*

Proof. The proof is similar to that of Theorem 1. If f is not a finite Blaschke product, then, for any g in H_∞,

$$\| z^n (1 - z)^k - f(z)\, g(z) \|_\infty \geq \delta,$$

where δ is a positive number depending only on k and f. This is so because f(z) tends to zero along a sequence of points that converge to a point z_o ($z_o \neq 1$, $|z_o| = 1$). We now proceed as in the proof of Theorem 1, except that we replace the numbers a_n by their k-th differences.

Remark. By the same method it becomes clear that any moving average of the coefficients of a nonrational inner function f can never be o(1/n), if we postulate that the inner function be small near some boundary point where the "characteristic function" of the moving average is not. For example, if $a_n - a_{n+2} = o(1/n)$, the only possible singularities of f on $|z| = 1$ are at ± 1.

2. COEFFICIENTS OF A SPECIAL FUNCTION

In this section we obtain estimates for the Taylor coefficients of a special inner function that will be needed in the sequel. Let

$$I_a(z) = \exp\left(a\,\frac{z+1}{z-1}\right) = \sum_{n=0}^{\infty} c_n z^n \qquad (a > 0).$$

The asymptotic behaviour of the c_n may be deduced fròm known results concerning confluent hypergeometric functions. (This was also noted by G. T. Cargo and by A. L. Shields).

We have the known identity [8, p. 100]

$$\exp\frac{-bz}{1-z} = \sum_{n=0}^{\infty} L_n^{(-1)}(b)\,z^n,$$

where the $L_n^{(-1)}$ denote generalized Laguerre polynomials. For $b = 2a$, this gives

$$c_n = e^{-a} L_n^{-1}(2a).$$

Applying a formula of Fejér (see [8, p. 196]), we obtain

$$c_n = \pi^{-1/2}(2a)^{1/4} n^{-3/4} \cos\left(2(2an)^{1/2} + \frac{\pi}{4}\right) + O(n^{-5/4}).$$

Thus, the c_n behave qualitatively like $n^{-3/4}\cos(n^{1/2})$. From (1) we deduce readily

$$\sum_{n \geq N} c_n^2 > BN^{-1/2}, \tag{2}$$

where B is a positive constant depending only on a.

3. FUNCTIONS WITHOUT ZEROS

THEOREM 4. *If* $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ *is a nonconstant inner function that does not vanish in* $|z| < 1$, *then*

$$\sum_{n \geq N} |a_n|^2 \geq BN^{-1/2},$$

where B *is a positive constant depending only on* a_0*. In particular,* a_n *cannot be* $o(n^{-3/4})$.

Proof. We may assume $f(0) > 0$. Let $g(z) = \log f(z)$, where that branch of the logarithm is chosen for which $g(0)$ is real. Let

$$\omega(z) = \frac{g(z) - g(0)}{g(z) + g(0)}. \tag{3}$$

Then $\omega(z)$ is regular in $|z| < 1$ and bounded by 1, and $\omega(0) = 0$. Solving (3), we obtain

$$g(z) = g(0)\,\frac{1 + \omega(z)}{1 - \omega(z)}.$$

Let $a = -\log a_o > 0$; then

$$f(z) = I_a[\omega(z)], \tag{4}$$

where

$$I_a(z) = \exp\left(a\,\frac{z + 1}{z - 1}\right).$$

Thus, $f(z)$ is *subordinate* to $I_a(z)$ in $|z| < 1$ (Littlewood [4, p. 163]) and hence [4, Theorem 215]

$$\sum_1^n |a_i|^2 \leq \sum_1^n c_i^2 \qquad (n \geq 1),$$

where the c_i are the Taylor coefficients of $I_a(z)$. Since $a_o = c_o$ and

$$\sum_0^\infty |a_i|^2 = \sum_0^\infty c_i^2 = 1,$$

we conclude that

$$\sum_{n+1}^\infty |a_i|^2 \geq \sum_{n+1}^\infty c_i^2.$$

The theorem now follows from inequality (2).

With the use of Theorem 214 of [4], we could also exploit the subordination relation (4) to obtain information about the zeros of $f(z) - \lambda$, where $|\lambda| < 1$.

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