A NEW APPROACH TO THE FIRST FUNDAMENTAL THEOREM ON VALUE DISTRIBUTION

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1. The fundamental theorems of R. Nevanlinna in the theory of functions meromorphic in the plane or unit circle [8] were stated in a slightly different form by Ahlfors [1] by means of a new technique. Using the methods of Ahlfors, the author extended the results to functions meromorphic in more general domains D [5]. Later, analogous though weaker results were derived for functions of pseudomeromorphic character in D [6]. These functions can be characterized as being quasiconformal in every compact subdomain of D and having continuous partial derivatives. In this connection, formulas of Ozaki, Ono, and Ozawa [9] were useful. They can be interpreted as an unintegrated form of the first fundamental theorem.

The aim of the present paper is to give a new proof of the unintegrated first fundamental theorem, more in the style of the Ahlfors methods, and to state it for a wider class of functions than the one considered by the original authors. Further, we give a somewhat better estimate of the remainder term occuring in the analogue of the integrated first fundamental theorem for pseudomeromorphic functions; this estimate is valid for a slightly wider class than that previously considered. The corresponding theorem for functions of meromorphic character is an easy corollary of a step in the deduction; but the new approach is of course more involved than the original concept used for this special case in [1] and [5].

2. Let D be an open domain in the z-plane $(z = x + iy)$, and Γ its boundary. Let D be exhausted by open subdomains Δ_λ $(-\infty < \lambda < \lambda_0)$ with boundaries $G_\lambda \subset D$, each consisting of a finite number of Jordan curves. We assume Δ_λ to be increasing with λ, so that $\Delta_\lambda \cup G_\lambda \subset \Delta_{\lambda'}$ for $\lambda < \lambda'$, and so that Δ_λ tends to D for $\lambda \nearrow \lambda_0$ and to some inner point O (which for convenience we take as $z = 0$) as $\lambda \searrow -\infty$.

3. Let the function

$$w(z) = u(x, y) + i\,v(x, y)$$

provide a mapping of D into the Riemann w-sphere W. We assume that

a) $w(z)$ is continuous (in the spherical sense);

b) there is no accumulation point of the roots of $w(z) = a$ in D, for any a; and

c) the mapping is sense-preserving in the following strong sense: Whenever a sufficiently small circle with center $z_0 \in D$ is described once in the positive sense, then $\arg[w(z) - w(z_0)]$ increases by $2\pi k$, where k is a positive integer (unless $w(z_0) = \infty$, in which case the obvious analogue on W has to be considered).

From c) it follows that k can be defined as the multiplicity of an a-point z_0. We see at once that *the principle of the argument* remains valid, so that for a domain Δ bounded by Jordan arcs $G \subset D$,

Received January 29, 1962.

$$\int_G d\{\arg[w(z) - a]\} = 2\pi[n(\Delta, a) - n(\Delta, \infty)]$$

provided $w(z) \neq a, \infty$ on G. Here $n(\Delta, b)$ denotes the number (multiplicity sum) of b-points in Δ.

A mapping w is called *interior* if it satisfies a) and b) and if further

c') it maps every open set onto an open set.

From c) and a) it follows that every interior point of an open set is mapped into an interior point of the image set, and therefore our mapping is interior. Conversely, if a mapping is interior and sense-preserving, then it also satisfies our conditions a), b), c) [12, p. 88].

4. For $w(z) \neq a$ on G_λ, we define

$$\mu(\lambda, a) = \frac{1}{2\pi}\int_{G_\lambda}\left\{\frac{|w|^2\, d(\arg w)}{1 + |w|^2} - d[\arg(w - a)]\right\} \qquad (a \neq \infty), \tag{1}$$

$$\mu(\lambda, \infty) = \frac{1}{2\pi}\int_{G_\lambda}\frac{|w|^2\, d(\arg w)}{1 + |w|^2}. \tag{2}$$

The argument principle then gives $\mu(\lambda, \infty) - \mu(\lambda, a) = n(\lambda, a) - n(\lambda, \infty)$, where $n(\lambda, b)$ means $n(\Delta_\lambda, b)$. Therefore

$$A(\lambda) = \mu(\lambda, a) + n(\lambda, a) \tag{3}$$

is independent of a. Also, for $a \neq \infty$,

$$\begin{aligned}\mu(\lambda, a) &= \frac{1}{2\pi}\,\Im\int_{G_\lambda}\left\{\frac{|w|^2}{1 + |w|^2}\, d\log w - d\log(w - a)\right\} \\ &= \frac{1}{2\pi}\,\Im\int_{G_\lambda}\left\{\frac{\bar{w}}{1 + |w|^2} - \frac{1}{w - a}\right\} dw.\end{aligned}$$

But

$$\Re\int_{G_\lambda}\frac{\bar{w}\, dw}{1 + |w|^2} = \int_{G_\lambda}\frac{u\, du + v\, dv}{1 + u^2 + v^2} = \frac{1}{2}\int_{G_\lambda} d\log(1 + u^2 + v^2) = 0, \quad \Re\int_{G_\lambda}\frac{dw}{w - a} = 0,$$

so that for $a \neq \infty$

$$\mu(\lambda, a) = \frac{1}{2\pi i}\int_{G_\lambda}\left\{\frac{\bar{w}}{1 + |w|^2} - \frac{1}{w - a}\right\} dw = -\frac{1}{2\pi i}\int_{G_\lambda}\frac{1 + a\bar{w}}{w - a}\,\frac{dw}{1 + |w|^2}, \tag{4}$$

and likewise

(5) $$\mu(\lambda, \infty) = \frac{1}{2\pi i} \int_{G_\lambda} \frac{\bar{w}\, dw}{1 + |w|^2}.$$

If we integrate $\mu(\lambda, a)$ over the Riemann sphere W, with respect to $a = \rho e^{i\psi}$, we have

$$\int_W \mu(\lambda, a)\, dW(a) = -\frac{1}{2\pi i} \int_{G_\lambda} \frac{dw}{1 + |w|^2} \int_0^\infty \int_0^{2\pi} \frac{1 + \rho e^{i\psi} \bar{w}}{w - \rho e^{i\psi}} \frac{\rho\, d\rho\, d\psi}{(1 + \rho^2)^2}$$

$$= i \int_{G_\lambda} \frac{dw}{1 + |w|^2} \int_0^\infty \frac{\rho\, d\rho}{(1 + \rho^2)^2} \cdot \frac{1}{2\pi i} \int_{|t|=1} \frac{1 + \rho \bar{w} t}{w - \rho t} \frac{dt}{t}.$$

The residue of the last integrand is

$$\frac{1}{w} \text{ at } t = 0 \quad \text{and} \quad -\frac{1 + |w|^2}{w} \text{ at } t = \frac{w}{\rho},$$

so that

$$\int_W \mu(\lambda, a)\, dW = i \int_{G_\lambda} \frac{dw}{1 + |w|^2} \left\{ \frac{1}{w} \int_0^{|w|} \frac{\rho\, d\rho}{(1 + \rho^2)^2} - \bar{w} \int_{|w|}^\infty \frac{\rho\, d\rho}{(1 + \rho^2)^2} \right\} = 0.$$

From (3) and $\int_W dW(a) = \pi$ it now follows that

$$A(\lambda) = \frac{1}{\pi} \int_W n(\lambda, a)\, dW(a);$$

that is, $A(\lambda)$ is identified as the *Shimizu characteristic*, πA being the area of the image surface of Δ_λ spread over W. Formula (3) with the expressions (1) and (2) or (4) and (5) for μ is then recognized as the *first fundamental theorem in unintegrated form*. It may also be noted that for $a \neq \infty$ we can find a form of μ corresponding to (2), namely

(6) $$\mu(\lambda, a) = \frac{1}{2\pi} \int_{G_\lambda} \frac{|w_a|^2}{1 + |w_a|^2} d(\arg w_a),$$

where w_a is the rotational transform of $w = w_\infty$:

$$w_a = \frac{1 + \bar{a} w}{w - a}.$$

5. The unintegrated form (3) of the first fundamental theorem as a statement in terms of z is essentially of topological character. In fact, all previous integrations could be carried out entirely in the w-plane. In this form, (3) occurs already in the paper [2, p. 7] of Ahlfors, with (see (13) below)

(6a)
$$\mu(\lambda, a) = \frac{1}{2\pi} \int_\gamma \frac{\partial}{\partial n} \log \frac{1}{[w, a]} \, ds,$$

where γ is the w-image of G_λ. In the integrated form (as in the original form by R. Nevanlinna [8] and in Ahlfors's modification [1]), the metric relation between w and the z-plane comes into view. To be able to maintain the definitions of the symbols occuring in this theory and to carry out the proofs, we therefore confine ourselves to a more restricted class of functions w(z). For the same reason, the method of exhaustion and the parameter λ associated with it must be specified. This also imposes certain restrictions on the domain D.

Let D again be normalized to contain the point $z = 0$. We require that there exist a harmonic function g(z) which has the development

$$g(z) = \log |z| + \text{a harmonic function}$$

in the neighborhood of 0 and is regular elsewhere in D. Furthermore g(z) shall converge uniformly to a limit λ_0 as z approaches the boundary Γ of D. We have two possible cases: a) the parabolic case, where Γ is a point set of logarithmic capacity 0, $\lambda_0 = +\infty$, and g(z) is a Selberg-function (compare [5] or [10]); b) the hyperbolic case, where D has a Greens function $-g(z)$ and Γ consists entirely of regular points, in the sense that $\lim g(z) = 0$ uniformly on Γ, so that we can take $\lambda_0 = 0$. In both cases, Δ_λ shall be the domain $g(z) < \lambda$, and G_λ the system of level curves $g(z) = \lambda$. Also, in both cases the harmonic conjugate h(z) of g is multiple-valued, but it increases by 2π as G_λ is described in its entirety with Δ_λ to the left. We consider the analytic function

$$s(z) = g(z) + i\,h(z).$$

In the classical case, where D is the whole z-plane or the circle $|z| < 1$, we have

$$s(z) = \log z, \quad g = \log r, \quad h = \phi \qquad (z = re^{i\phi}).$$

6. As to the function $w(z) = u + iv$ defined in D, we now assume that

A) w(z) is continuous, and the partial derivatives u_x, u_y, v_x, v_y exist and are continuous;

B) an inverse Hölder condition is satisfied; in other words, to each compact subdomain of D there corresponds a finite κ such that

(7)
$$|w(z) - w(z_0)| \geq |z - z_0|^\kappa$$

whenever $|z - z_0| \leq \delta$, δ being a suitable positive number;

C) $J(z) = \dfrac{\partial(u, v)}{\partial(x, y)} > 0$, except possibly on a point set having no limit points in D.

We shall allow w(z) to be infinite at points of D; at such points and in neighborhoods of them, $1/w$ instead of w shall fulfill the above conditions.

Conditions a), b), and c) of Section 3 are consequences of A), B), and C), respectively. We can therefore apply our previous results to the more restricted class of functions defined here.

7. At points where $J(z) > 0$, the dilatation quotient $Q(z)$ can be computed from

$$Q(z) + 1/Q(z) = (u_x^2 + u_y^2 + v_x^2 + v_y^2)/J(z). \tag{8}$$

The mean derivative $\partial w/\partial z$ and the Pompéiu derivative $\partial w/\partial \bar{z}$ are (see [3, p. 41], [7], or [11])

$$\frac{\partial w}{\partial z} = \frac{1}{2}[u_x + v_y + i(v_x - u_y)], \tag{9}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}[u_x - v_y + i(v_x + u_y)]. \tag{10}$$

We know $Q(z)$ to be invariant with respect to conformal transformations, so that if $f(w)$ and $z(\zeta)$ are analytic functions, then $f\{w[z(\zeta)]\}$ and $w(z)$ have the same value Q at corresponding points ζ and $z(\zeta)$. It is easv to show that in this case $\partial w/\partial z$ and $\partial w/\partial \bar{z}$ are transformed by means of the formulas

$$\frac{\partial f}{\partial \zeta} = f'(w)\frac{\partial w}{\partial z}z'(\zeta) \quad \text{and} \quad \frac{\partial f}{\partial \bar{\zeta}} = f'(w)\frac{\partial w}{\partial \bar{z}}\overline{z'(\zeta)}. \tag{11}$$

Also, from (8) to (10) we get

$$\left|\frac{\partial w}{\partial z}\right| = \frac{1}{2}(Q + 1)\sqrt{\frac{J}{Q}} \quad \text{and} \quad \left|\frac{\partial w}{\partial \bar{z}}\right| = \frac{1}{2}(Q - 1)\sqrt{\frac{J}{Q}}. \tag{12}$$

We can combine conditions A) and C) of Section 6 with the condition

B_1) $Q(z) \leq K_\triangle < \infty$ in every compact subdomain $\triangle$ of D (in other words, $w(z)$ is quasiconformal in $\triangle$).

A function $w(z)$ satisfying these conditions is said to be *pseudomeromorphic* in D (see [6] and [11]). In $\triangle$, the pseudomeromorphic function $w(z)$ can be considered as a composite function $w[\zeta(z)]$, where $w(\zeta)$ is meromorphic and $\zeta(z)$ is a quasiconformal homeomorphism. This can be shown in the following way. For each $z \in \triangle$ (with exception of the finite point set E where $J = 0$), $w(z)$ determines $Q(z)$ and, wherever $Q(z) > 1$, also an angle $\psi(z)$ that can be described as follows: $\psi(z)$ is the angle between the real axis in the z-plane and the preimage of the major axis of the infinitesimal ellipse onto which $w(z)$ maps the infinitesimal circle with center z. As was shown in [7], it is possible to construct a quasiconformal mapping $\zeta(z)$ of $\triangle$ with these preassigned values $Q(z)$ and $\psi(z)$. The function $w(z(\zeta))$ then maps the domain $\zeta(\triangle)$ in such a way that in each point of $\zeta(\triangle - E)$ where $Q = 1$, infinitesimal circles are carried onto infinitesimal circles, and in the other points infinitesimal ellipses are carried onto similar infinitesimal ellipses, major axes being mapped onto major axes. But then the map is conformal and sense-preserving, and $w(\zeta)$ is of meromorphic character. Because z as a function of ζ is subject to a Hölder condition [4], (7) is valid for $\zeta(z)$, and therefore, with appropriately changed exponent it holds also for $w(z)$. Therefore *the pseudomeromorphic functions belong to the class considered.* Because the mapping $w = x^3 + 3xy^2 + iy$ is not quasiconformal in any $\triangle$ containing 0, but satisfies conditions A) to C), the classes do not coincide. (In [11], however, all functions satisfying A) and C) are called pseudomeromorphic.)

8. Let $[w, a]$ denote the chordal distance between w and a on W. Then

$$(13)\quad \begin{cases} [w, a] = (1 + |w_a|^2)^{-1/2} = |w - a|(1 + |w|^2)^{-1/2}(1 + |a|^2)^{-1/2} & (a \neq \infty), \\ [w, \infty] = (1 + |w|^2)^{-1/2}. \end{cases}$$

Just as for functions meromorphic in D, we can, for the class considered in Section 6, define

$$(14)\quad m(\lambda, a) = \frac{1}{2\pi}\int_{G_\lambda} \log \frac{1}{[w(z), a]}\, dh,$$

$$N(\lambda, a) = \int_{-\infty}^{\lambda} n(g, a)\, dg + \log[w(0), a],$$

where the formula for $N(\lambda, a)$ should be slightly modified for $a = w(0)$. Condition B) guarantees the convergence of (14), and we have, for all a,

$$(15)\quad \lim_{\lambda \to -\infty} [m(\lambda, a) + N(\lambda, a)] = 0.$$

The Nevanlinna-Ahlfors characteristic is

$$T(\lambda) = \int_{-\infty}^{\lambda} A(g)\, dg.$$

Integration of (3) then gives [for $a \neq w(0)$]

$$(16)\quad m(\lambda, a) + N(\lambda, a) = T(\lambda) + E(\lambda, a),$$

where, by (6) and (13) to (15),

$$(17)\quad \begin{aligned} E(\lambda, a) &= m(\lambda, a) - m(-\infty, a) - \int_{-\infty}^{\lambda} \mu(\lambda, a)\, d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\lambda} d\lambda \int_{G_\lambda} \frac{|w_a|^2}{1 + |w_a|^2}\left\{\frac{\partial(\log|w_a|)}{\partial\lambda} - \frac{\partial(\arg w_a)}{\partial h}\right\} dh. \end{aligned}$$

If in particular $w(z)$ is of *meromorphic* character, then $\log w_a$ is an analytic function of $s(z)$, and the Cauchy-Riemann equations yield $E(\lambda, a) \equiv 0$. Therefore the *first fundamental theorem* is valid in the form (see [5, p. 21])

$$(18)\quad m(\lambda, a) + N(\lambda, a) = T(\lambda).$$

Also, in this case the invariance of the operation $\frac{\partial}{\partial n} ds$ in (6a) immediately shows that $\mu(\lambda, a) = \frac{dm(\lambda, a)}{d\lambda}$, so that (18) holds.

In the *general* case, however, (16) *constitutes the modified form* of the *first fundamental theorem.* It remains to compute or estimate the term $E(\lambda, a)$. It turns out (see Section 10 below) that E can vanish identically for all a, even for a function with $Q(z)$ unbounded and increasing arbitrarily fast with λ. On the other hand $E(\lambda, a)$ may be of the same order of growth as $T(\lambda)$, even in cases where $Q(z)$ is bounded.

9. From (10), (11), and (17) we get

$$E(\lambda, a) = \frac{1}{\pi}\iint_{\Delta_\lambda} \Re \frac{\partial(\log w_a)}{d\bar{s}} \frac{|w_a|^2 dg\, dh}{1 + |w_a|^2}$$

$$= \frac{1}{\pi}\iint_{\Delta_\lambda} \Re \left\{\frac{1}{w_a}\frac{dw_a}{dw}\frac{\partial w}{\partial \bar{z}}\frac{d\bar{z}}{ds}\right\}\frac{|w_a|^2 dg\, dh}{1 + |w_a|^2}$$

$$= \frac{1}{\pi}\iint_{\Delta_\lambda} \Re \left\{\frac{dw_a}{dw}\frac{\partial w}{\partial \bar{z}}\frac{ds}{dz}\bar{w}_a\right\}\frac{dx\, dy}{1 + |w_a|^2}.$$

From this we compute

$$E(\lambda, a) = -\frac{1}{\pi}\iint_{\Delta_\lambda} \Re \left\{\frac{1 + a\bar{w}}{w - a}\frac{\partial w}{\partial \bar{z}}\frac{ds}{dz}\right\}\frac{dx\, dy}{1 + |w|^2} \qquad (a \neq \infty),$$

$$E(\lambda, \infty) = \frac{1}{\pi}\iint_{\Delta_\lambda} \Re \left\{\bar{w}\frac{\partial w}{\partial \bar{z}}\frac{ds}{dz}\right\}\frac{dx\, dy}{1 + |w|^2}.$$

Now (12) gives the estimate

$$|E(\lambda, a)| \leq \frac{1}{2\pi}\iint_{\Delta_\lambda} [Q(z) - 1]\sqrt{\frac{J(z)}{Q(z)}}\left|\frac{ds}{dz}\right|\frac{|w_a|\, dx\, dy}{1 + |w(z)|^2}, \tag{19}$$

which, in the case where D is a circle or the whole plane, reads

$$|E(\log r, a)| \leq \frac{1}{2\pi}\iint_{|z|<r} [Q(z) - 1]\sqrt{\frac{J(z)}{Q(z)}}\frac{|w_a|\, dr\, d\phi}{1 + |w(z)|^2}. \tag{20}$$

These results provide a slight improvement over the previous estimate (20) in [6].

10. In the following examples, take D as the z-plane or a circle, and $w(z) = f[\zeta(z)]$, where $f(\zeta)$ is meromorphic and $\zeta(z)$ is a quasiconformal homeomorphism. We set $z = re^{i\phi}$, $\zeta = \rho e^{i\psi}$.

Example 1. Set $\rho = r$, $\psi = \phi + s(r)$, where $s(0) = s'(0) = 0$, and where $s(r) > 0$ and $s'(r) > 0$ for $r > 0$. Then $Q(re^{i\phi}) > r^2 s'(r)^2$. If $f(\zeta)$ and a finite set of values a are given, we can therefore choose $s'(r)$ growing rapidly enough so that the right-hand side of (20) increases more rapidly than $T(r)$ or any arbitrary function tending to infinity as $r \to 1$ or $r \to \infty$, respectively. But in fact $E(\log r, a) \equiv 0$ for every a, as can be seen from (17) or from the fact that $\zeta(z)$ only twists the plane, so that obviously the values of m, N, and T are the same for $w(z)$ as for $f(\zeta)$. Therefore (18) is valid.

Example 2. Set $\rho = \rho(r)$, $\psi = \phi$, with $\lim_{r\to 0} \rho/r = 1$ and $\rho \nearrow \infty$ as $r \nearrow \infty$, and let $f(\zeta) = \zeta^n$. Then $T(r) = n \log r\,[1 - o(1)]$ for $r \to \infty$. For $a \neq \infty$, $E(\log r, a)$ is bounded, so that (18) holds if only a bounded term is added. If, for $a = \infty$, $\mathfrak{R}\,\frac{\partial(\log w)}{\partial \bar{s}}$ does not change sign, then the right-hand side of (20) gives the exact value of $|E(\log r, \infty)|$, because $\mathfrak{I}\,\frac{\partial(\log w)}{\partial \bar{s}} \equiv 0$. This is true when $r\rho' - \rho$ is nonnegative or nonpositive for all r. 2α) The first case is valid for $\rho(r) = r + r^k$ with $k > 1$. Here $E(\log r, \infty) = (k - 1)\,n \log r + O(1)$, so that $\lim_{r\to\infty} E/T = k - 1 > 0$. Also, $Q(z) \to k$ as $r \to \infty$. 2β) The second case occurs if $\log \rho = \log r - k \arctan r$ and $0 < k < 2$. Then also $E(\log r, \infty)$ is bounded, and the first fundamental theorem holds with a bounded remainder term. This phenomenon is to be expected whenever the mapping is nearly conformal in the sense that, as z becomes large, $Q(z) \to 1$ so rapidly that the right-hand side of (20) remains bounded.

REFERENCES

1. L. Ahlfors, *Über eine Methode in der Theorie der meromorphen Funktionen,* Soc. Sci. Fenn. Comment. Phys.-Math. 8, No. 10 (1935), 1-14.

2. ———, *Über die Anwendung differentialgeometrischer Methoden zur Untersuchung von Überlagerungsflächen,* Acta Soc. Sci. Fenn., N.S.A., 2, No. 6 (1937), 1-17.

3. ———, *Complex analysis,* McGraw-Hill, New York, 1953.

4. R. Finn and J. Serrin, *On the Hölder continuity of quasi-conformal and elliptic mappings,* Trans. Amer. Math. Soc. 89 (1958), 1-15.

5. G. af Hällström, *Über meromorphe Funktionen mit mehrfach zusammenhängenden Existenzgebieten,* Acta Acad. Aboensis, 12, No. 8 (1939), 1-100.

6. ———, *Wertverteilungssätze pseudomeromorpher Funktionen,* Acta. Acad. Aboensis, 21, No. 9 (1958), 1-23.

7. O. Lehto and K. I. Virtanen, *On the existence of quasiconformal mappings with prescribed complex dilatation,* Ann. Acad. Sci. Fenn. Ser. A. I. 274, (1960), 1-24.

8. R. Nevanlinna, *Eindeutige analytische Funktionen,* Springer, Berlin, 1936 and 1953.

9. S. Ozaki, I. Ono, and M. Ozawa, *On the function-theoretic identities, I,* Sci. Rep., Tokyo Bunrika Daigaku, Sect. A. 4, (1951), 157-160.

10. H. L. Selberg, *Über die ebenen Punktmengen von der Kapazität Null,* Avh. Norske Vid. Akad. Oslo. I No. 10 (1937), 1-10.

11. Y. Tôki and K. Shibata, *On the pseudo-analytic functions,* Osaka Math. J. 6 (1954), 145-165.

12. G. T. Whyburn, *Topological analysis,* Princeton Mathematical Series No. 23, Princeton Univ. Press, 1958.

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