FACTORS OF N-SPACE

Kyung Whan Kwun

1. INTRODUCTION

Bing [2] showed that a certain locally bad 3-gm is a cartesian factor of E^4. Curtis and Wilder [5] showed that the space of Bing, although pathological, is nevertheless locally like E^3 in the sense of homotopy. Raymond [8] proved that every 3-dimensional cartesian factor of E^4 is necessarily locally like E^3 in the sense of homology. Later, Rosen [9] used Bing's construction to show that there exists a nowhere euclidean cartesian factor of E^4. However, it follows easily from our result [7] that his space is a homotopy manifold. It was Curtis [4] who first showed that there exists a cartesian factor of E^4 that is not a homotopy manifold, and who thus answered in the negative a question raised in the original draft of [7]. By constructing a certain pseudo-isotopy of E^{n+1}, by the methods of [2], Andrews and Curtis [1] recently showed that if one shrinks an arc in E^n to a point and then multiplies by the line, then the resulting space is E^{n+1}. In view of [10], this proposition enables us to obtain results similar to those of [9], for all dimensions greater than 2. Furthermore, we can construct the space so that no open subset of it is locally like E^n in the sense of homotopy, and we can replace the construction and argument of [9] by simpler ones. In particular, our construction is similar to one in our earlier work [6]. We also remark that the technique of the present work gives the affirmative answer to a question raised in [6] with the proviso that the construction should be careful.

2. A CERTAIN ARC IN E^n

The following lemma provides us with an arc that we shall use later.

LEMMA 1. *For each* $n \geq 3$, *there exists an arc* P *in* E^n *such that for each open set* U *containing* P *there exists a simple closed curve* C *in* U - P *which is not deformable to a point (that is, whose inclusion map is not null-homotopic) in* E^n - P.

Remark. The arc that we shall use must have a property much stronger than non-simple connectedness of the complement. In the following proof of Lemma 1, we assume the reader's familiarity with the construction of Blankinship [3]. The proof mainly describes what particular set of circles should be avoided in constructing the n-cell E of Blankinship. We use the notation of [3].

Proof of Lemma 1. Let y be the simple closed curve on Bd T that is not deformable to a point in E^n - A. Let y_α be the image of y under the global homeomorphism f_α, where $\alpha = i_1 i_2 \cdots i_j$ $(i_p \leq k)$ denotes any array of appropriate positive integers, and $f_\alpha = f_{i_1} f_{i_2} \cdots f_{i_j}$ as in [3]. Let Y be the sum of the sets y_α. We obtain an arc as described in Lemma 1 by avoiding Y in constructing Blankinship's n-cell E and then applying his method.

Received October 25, 1961.

The present work was supported in part by the Air Force Contract AF 49(638)-774 through the University of Michigan.

Throughout this paper, the set-theoretic interior and the boundary of a set A in a space will be denoted by int A and bdy A, respectively, while the symbols Int A and Bd A are reserved to denote the interior and the boundary of a manifold A with a boundary.

For later use, we formulate the result of Andrews and Curtis as follows.

THEOREM 2 (Andrews and Curtis). *Let α be any arc in E^n. Then there exists a sequence of compact neighborhoods $T_0, T_1, T_2, \cdots$ of α such that $T_i \subset \text{int } T_{i-1}$ for each i, such that $T_0 \cdot T_1 \cdot T_2 \cdots = \alpha$, and such that the following condition is satisfied:*

To each $\varepsilon > 0$ and each positive integer i there correspond a positive integer N and a uniformly continuous isotopy μ of E^{n+1} onto itself such that

(1) *μ_0 is the identity,*

(2) *each μ_t is the identity outside of $T_i \times E^1$,*

(3) *μ_t moves no point of E^{n+1} along the w-direction as much as ε, where w denotes the (n+1)st coordinates of the points of E^{n+1}, and*

(4) *for each w, the diameter of $\mu_1(T_N \times w)$ is less than ε.*

We shall construct an upper-semicontinuous decomposition G of E^n such that the decomposition space X has the desired property.

3. CONSTRUCTION OF G

Let α be the arc P of Lemma 1, and $T_0, T_1, \cdots$ a sequence of compact neighborhoods of α as in Theorem 2. Now let h be a homeomorphism of E^n onto itself. Then Theorem 2 holds, with $h(\alpha)$ and $h(T_i)$ replacing α and T_i, respectively, if we take a sufficiently large N. This is true because h and the identity of E^1 induce a global homeomorphism of E^{n+1} and h is uniformly continuous in T_0. Let K be a tame n-cell in E^n such that $\alpha \subset \text{Int } K$ (an n-cell D in E^n is called *tame* if $\text{Cl}(E^n - D)$ is homeomorphic to an n-cell minus an interior point). If K' is another tame n-cell and h is a homeomorphism of K onto K', there exists an extension homeomorphism H of h such that E^n is mapped onto itself under H. Hence $H(\alpha)$ in Int K' satisfies Theorem 2. This fact will be used later. Throughout the paper, we use α, T_i, and K exclusively for the sets defined here.

1. Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers, converging to 0, whose terms are yet to be determined.

2. For each positive integer m, let F_m denote the compact subset of E^n consisting of the points x with $\|x\| \leq m$. Then $E^n = \Sigma F_m$.

3. For each integer m, let $\mathcal{U}^m = \{U_{m1}, U_{m2}, \cdots, U_{mk_m}\}$ denote a collection of open subsets of E^n such that each U_{mi} is the interior of a tame n-cell K_{mi}, $F_m \subset \Sigma U_{mi}$, and each U_{mi} is of diameter less than δ_m.

4. Now we want to find homeomorphisms h_{pq} of K onto K_{pq}. We construct $h_{p1}, h_{p2}, \cdots, h_{pk_p}$ simultaneously after all homeomorphisms $h_{p'q}$ $(p' < p)$ have been constructed. For each positive integer m and each integer $k \leq k_m$, we construct an h_{mk} such that the $h_{mk}(T_0)$ (m fixed and $k = 1, 2, \cdots, k_m$) are mutually disjoint and lie outside of the set

$$R_m = \sum_{p=1}^{m-1} \sum_{q=1}^{k_p} \sum_{r=1}^{\infty} [h_{pq}(\text{bdy } T_r) + h_{pq}(\alpha) + \text{Bd } K_{pq}] + \sum_t \text{Bd } K_{mt}.$$

This is possible, since R_m is a nowhere dense compact subset of E^n.

5. Our G consists of $h_{pq}(\alpha)$ and the points not on any of these arcs.

4. PROPERTIES OF G

THEOREM 3. G *is upper-semicontinuous, and the decomposition space* X *is finite-dimensional and contains no open subset that is a homotopy* n-*manifold.*

Proof. A subset of E^n is called *saturated* (with respect to G) if it is the sum of elements of G. The upper-semicontinuity of G and the finite-dimensionality of X are simultaneously proved by showing that each element of G has an arbitrarily close neighborhood U such that U is saturated and bdy U is the sum of degenerate elements of G. If $g \in G$ is an $h_{pq}(\alpha)$, then an $h_{pq}(\text{int } T_r)$ with a sufficiently large r is such a U. If g is a point, let $U_{m,k(g)}$ be an element of $\mathscr{U}^m$ containing g. Then

$$U_{m,k(g)} - \sum_{p=1}^{m-1} \sum_{q=1}^{k_p} h_{pq}(T_M),$$

with sufficiently large m and M, is such a U. Here of course M depends on m.

To prove the last part, suppose V is an open subset of X that is a homotopy manifold. Then for each $x \in V$, there exists an open set V' containing x and such that every loop in V' - x is nullhomotopic. By a theorem of Smale [10], this means that if $U = f^{-1}(V)$, f is the quotient map of G, and g is an element of G in U, then g has a sufficiently close neighborhood U' such that every simple closed curve in U' - g is deformable to a point in U - g. But since the totality of the $\mathscr{U}^m$ is a base for the open sets of E^n, there exists some $h_{pq}(\alpha)$ in U. Hence we bring about a situation that is contradictory to our construction and Lemma 1.

5. CHOICES OF δ_m AND ISOTOPIES IN E^{n+1}

To prove that $X \times E^1 = E^{n+1}$, at least for careful choices of δ_m, we alternate between describing $\mathscr{U}^m$ and describing isotopies in E^{n+1} (compare [9]). Let $\varepsilon_1, \varepsilon_2, \cdots$ be a decreasing sequence of positive numbers with a finite sum. We shall require that $\delta_m \leq \varepsilon_{m-1}$ for $m = 2, 3, \cdots$. Let $\delta_1 = 1$ and construct $\mathscr{U}^1$ and h_{1k}.

(1 - 1) By Theorem 2, there exist a uniformly continuous isotopy f_t $(0 \leq t \leq 1/2)$ of E^{n+1} onto itself, and a positive integer N_1, such that

1 - 1 - 1. f_0 is the identity,

1 - 1 - 2. f_t is the identity outside of $\Sigma h_{1i}(T_0) \times E^1$,

1 - 1 - 3. f_t moves no point of E^{n+1} along the w-direction as much as ε_1, and

1 - 1 - 4. for each w in E^1, the diameter of $f_{1/2}(h_{1i}(T_{N_1}) \times w)$ is less than ε_1.

(1 - 2) Let δ_2 be a positive number such that if D is any set in E^{n+1} of diameter less than δ_2, then $f_{1/2}(D)$ is a set of diameter less than ε_2 and $f_{1/2}$(any

δ_2-neighborhood of D) is an ε_2-neighborhood of $f_{1/2}(D)$. Using this δ_2, construct $\mathcal{U}^2$ and h_{2k}.

(2 - 1) By Theorem 2, there again exist a uniformly continuous isotopy h_t $(0 \leq t \leq 1/4)$ of E^{n+1} onto itself and an integer $N_2 > N_1$ such that

(1) h_0 is the identity,

(2) h_t is the identity outside of

$$\sum h_{1i}(T_{N_1}) \times E^1 + \sum h_{2i}(T_{N_1}) \times E^1 ,$$

(3) h_t moves no point of E^{n+1} along the w-direction as much as δ_2, and

(4) for each w in E^1, the diameter of $h_{1/4}(h_{mi}(T_{N_2}) \times w)$ $(m = 1, 2)$ is less than δ_2.

Then $f_t = f_{1/2} \cdot h_{t-1/2}$ $(1/2 \leq t \leq 3/4)$ is a uniformly continuous isotopy of E^{n+1} onto itself such that

2 - 1 - 2. $f_t = f_{1/2}$ outside of

$$\sum h_{1i}(T_{N_1}) \times E^1 + \sum h_{2i}(T_{N_1}) \times E^1 ,$$

2 - 1 - 3. $f_{1/2}(E^n \times [w - \varepsilon_2, w + \varepsilon_2]) \supset f_{3/4}(E^n \times w)$ for each $w \in E^1$, and

2 - 1 - 4. for each w in E^1, the diameter of $f_{3/4}(h_{mi}(T_{N_2}) \times w)$ $(m = 1, 2)$ is less than ε_2.

Continuing in this manner, we find a sequence $\{\delta_m\}$, an increasing sequence of positive integers N_m, and isotopies f_t $(1 - 2^{1-m} \leq t \leq 1 - 2^{-m})$ such that

m - 1 - 2. $f_t = f_{1-2^{1-m}}$ outside of

$$\sum_{p=1}^{m} \sum_{q=1}^{k_p} h_{pq}(T_{N_{m-1}}) \times E^1 ,$$

m - 1 - 3. $f_{1-2^{1-m}}(E^n \times [w - \varepsilon_m, w + \varepsilon_m]) \supset f_{1-2^{-m}}(E^n \times w)$ for each $w \in E^1$, and

m - 1 - 4. $f_{1-2^{-m}}(h_{pq}(T_{N_m}) \times w)$ is of diameter less than ε_m for $p \leq m$.

We let $f = \lim_p f_{1-2^{-p}}$. That f is a continuous map of E^{n+1} onto itself, sending each $g \times w$ $(g \in G,\ w \in E^1)$ to a point, is verified as in [2]. Using m - 1 - 3, we can also verify that $g \times w$ and $g' \times w'$ are mapped into distinct points under f, if $w \neq w'$. We finally want to show that $g \times w$ and $g' \times w$ are mapped into distinct points if $g \neq g'$. This is clear if some $h_{pq}(\text{bdy } T_r)$ separates g and g' in E^n. Otherwise, each of g and g' is contained in only a finite number of $h_{pq}(\text{int } T_r)$. Hence there exists an integer N such that if $m > N$, then

$$f_{1-2^{-m}}(g \times w) = f(g \times w) \qquad \text{and} \qquad f_{1-2^{-m}}(g' \times w) = f(g' \times w) .$$

This proves that $X \times E^1 = E^{n+1}$. Combining this result with Theorem 3, we have

THEOREM 4. *For each* $n \geq 3$, *there exists an* n-*dimensional generalized manifold* X *such that* $X \times E^1 = E^{n+1}$ *while no open subset* U *of* X *is a homotopy* n-*manifold* (in particular, U is not an open n-cell).

Proof. That X is a generalized n-manifold follows from [8] or [11].

If we compactify E^n by adding a point $\bar{p}$, then $G + \bar{p}$ is an upper-semicontinuous decomposition of S^n. We observe that during the isotopies described above, $\bar{p} \times E^1$ remains pointwise fixed. Hence,

THEOREM 5. *Theorem* 4 *is true with* E^{n+1} *replaced by* $S^n \times E^1$.

Similar theorems and results about the fixed point sets of involutions of certain spaces may also be obtained as in [2] and [9].

REFERENCES

1. J. J. Andrews and M. L. Curtis, n-*space modulo an arc*, Ann. of Math. (2) 75 (1962), 1-7.

2. R. H. Bing, *The cartesian product of a certain nonmanifold and a line is* E^4, Ann. of Math. (2) 70 (1959), 399-412.

3. W. A. Blankinship, *Generalization of a construction of Antoine*, Ann. of Math. (2) 53 (1951), 276-297.

4. M. L. Curtis, *Corollary to a proof due to Bing*, Abstract No. 571-15, Notices Amer. Math. Soc. 7 (1960), 482-483.

5. M. L. Curtis and R. L. Wilder, *The existence of certain types of manifolds*, Trans. Amer. Math. Soc. 91 (1959), 152-160.

6. K. W. Kwun, *A generalized manifold*, Michigan Math. J. 6 (1959), 299-302.

7. ———, *A fundamental theorem on decompositions of the sphere into points and tame arcs*, Proc. Amer. Math. Soc. 12 (1961), 47-50.

8. F. A. Raymond, *Separation and union theorems for generalized manifolds with boundary*, Michigan Math. J. 7 (1960), 7-21.

9. R. H. Rosen, E^4 *is the cartesian product of a totally non-euclidean space and* E^1, Ann. of Math. (2) 73 (1961), 349-361.

10. S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. 8 (1957), 604-610.

11. R. L. Wilder, *Monotone mappings of manifolds, II*, Michigan Math. J. 5 (1958), 19-23.

The Seoul National University, Seoul, Korea
The University of Michigan and University of Wisconsin