ISOMETRIC IMMERSION OF FLAT RIEMANNIAN MANIFOLDS IN EUCLIDEAN SPACE

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1. INTRODUCTION

Let $\psi: M \to \overline{M}$ be an isometric immersion of Riemannian manifolds. If z is a tangent vector of $\overline{M}$, orthogonal to $d\psi(M_m)$, there is a classically defined second fundamental form operator S_z on the tangent space M_m. Following [1], we express the same information about ψ by associating with each vector $x \in M_m$ a linear operator T_x on $\overline{M}_{\psi(m)}$, called the *difference operator* of x. The function T is characterized by the fact that each T_x is skew-symmetric and $T_x(z) = d\psi(S_z(x))$ for $x \in M_m$, where z has the same meaning as above. The symmetry of S_z is equivalent to the relation $T_x(d\psi(y)) = T_y(d\psi(x))$ for $x, y \in M_m$. If $m \in M$, let $\mathcal{N}(m)$ be the subspace of M_m consisting of all vectors x such that $T_x = 0$, and let $\nu(m)$ be the dimension of $\mathcal{N}(m)$. Chern and Kuiper [2] call this integer the *index of relative nullity of* ψ *at* m. We denote by n the minimum value of the function ν on M. Finally let $\mathcal{N}^\perp(m)$ be the orthogonal complement of $\mathcal{N}(m)$ in M_m.

We shall deal with the immersion $\psi: M^d \to R^{d+k}$ of a flat d-dimensional Riemannian manifold in (d + k)-dimensional Euclidean space. In this case the proof of Theorem 2 of [2] implies that *for each point* $m \in M$ *there exists a vector* $x \in \mathcal{N}^\perp(m)$ *such that* T_x *is one-to-one on* $d\psi(\mathcal{N}^\perp(m))$. Since the latter subspace has dimension $d - \nu(m)$, it follows that $k \geq d - \nu(m)$, so that the minimum relative nullity n of ψ is at least d - k. We shall prove

THEOREM 1. *Let* $\psi: M^d \to R^{d+k}$ *be an isometric immersion of a complete flat Riemannian manifold in Euclidean space. Then* M^d *contains a totally geodesic submanifold that is carried isometrically onto an entire* n*-dimensional plane in* R^{d+k}, *where* n *is the minimum relative nullity of* ψ.

The theorem is trivially true if n is zero, but since $n \geq d - k$ we can force n to be positive:

COROLLARY 1. *If the hypotheses of Theorem* 1 *are satisfied and* $k < d$, *then the image of* ψ *contains a* (d - k)*-dimensional plane in* R^{d+k}.

This implies the fundamental result of Tompkins [4] that a compact flat M^d cannot be isometrically immersed in R^{2d-1}. More generally, we have

COROLLARY 2. *A complete flat Riemannian manifold* M^d *does not have a bounded isometric immersion in* R^{2d-1}.

As with Tompkins' theorem, restrictions on dimension cannot here be weakened, for R^d has bounded imbeddings in R^{2d}, indeed, imbeddings whose images are as small as one likes: imbed R^1 as, say, a small spiral in R^2, then take the d-fold Riemannian product.

For k = 1, that is, for the case of a hypersurface, Hartman and Nirenberg have proved (Theorem III of [3]) that an isometric immersion of a complete flat M^d in R^{d+1} is cylindrical. In Theorem 2 we give a sufficient condition for such immersions to be cylindrical when $k > 1$.

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2. PROOF OF THEOREM 1

We deal throughout with a fixed immersion $\psi\colon M^d \to R^{d+k}$, which, when it seems safe to do so, we omit from the notation. For example, we simply write the above-mentioned symmetry property of the difference operators as $T_x(y) = T_y(x)$, where $x, y \in M_m$. Let G be the set of points of M on which the relative nullity takes its minimum value n. Then G is an open set of M, and $\mathcal{N}$ is a differentiable field of n-planes on G. To prove theorem 1, we shall show that $\mathcal{N}$ is completely integrable and the leaves of $\mathcal{N}$ are complete and totally geodesic, relative to ψ, in R^{d+k}. A differentiable field e of orthonormal $(d + k)$-frames defined on an open set of G is *adapted to* ψ if, for each point m in its domain, $e_1, \cdots, e_n$ provides a basis for $d\psi(\mathcal{N}(m))$, $e_{n+1}, \cdots, e_d$ a basis for $d\psi(\mathcal{N}^\perp(m))$, and $e_{d+1}, \cdots, e_{d+k}$ a basis for $(d\psi(M_m))^\perp$ in $(R^{d+k})_{\psi(m)}$. We adopt the index conventions

$$1 \leq a, b \leq n; \quad n + 1 \leq r, s \leq d; \quad 1 \leq i, j \leq d; \quad d + 1 \leq \alpha, \beta \leq d + k.$$

Let us exclude the trivial cases $n = 0$, $n = d$ of Theorem 1; then none of the categories above is empty. A frame field such as e is a differentiable mapping into the frame bundle F of R^{d+k}. Pulling the Euclidean connection form $\bar{\phi}$ of F down to G by way of e, we get

$$\phi_{ij} = \bar{\phi}_{ij} \circ de \qquad \text{(connection forms of M)},$$

$$\tau_{i\alpha} = \bar{\phi}_{i\alpha} \circ de \qquad \text{(Codazzi forms)},$$

$$\theta_{\alpha\beta} = \bar{\phi}_{\alpha\beta} \circ de \qquad \text{(normal connection forms)}.$$

The second structural equation on F then yields the second structural, Codazzi, and Ricci (Koehne) equations for the frame field e. For the difference operators we have $T_{e_i}(e_j) = \Sigma\tau_{\alpha j}(e_i)e_\alpha$; hence the symmetry property of T is equivalent to $\tau_{\alpha j}(e_i) = \tau_{\alpha i}(e_j)$. Thus from the definition of $\mathcal{N}$ we derive

(1) $$\tau_{\alpha a} = 0, \qquad \tau_{\alpha r}(e_a) = 0.$$

The forms $\tau_{\alpha r}$ describe $\mathcal{N}$ (on the domain of e) in the sense that

$$\mathcal{N}(m) = \{x \in M_m \mid \tau_{\alpha r}(x) = 0\}.$$

Thus the integrability of $\mathcal{N}$ follows, by the Frobenius theorem, from the Codazzi equations for $\tau_{r\alpha}$, which reduce to

(2) $$d\tau_{r\alpha} = -\sum \phi_{rs} \wedge \tau_{s\alpha} - \sum \tau_{r\beta} \wedge \theta_{\beta\alpha}.$$

To prove that the leaves of $\mathcal{N}$ are totally geodesic in R^{d+k}, it suffices, by the definition of $\mathcal{N}$, to prove they are totally geodesic in M. In fact, let α be a geodesic of a leaf L of $\mathcal{N}$. If L is totally geodesic in M, then α is also a geodesic of M, that is, it has acceleration $\alpha'' = 0$ when considered as a curve in M. But the velocity α' of α is always contained in $\mathcal{N}$, hence $T_{\alpha'} = 0$. Thus the general formula

$$(\psi \circ \alpha)'' = T_{\alpha'}(d\psi(\alpha')) + d\psi(\alpha'')$$

shows that $(\psi \circ \alpha)'' = 0$. We conclude that the immersion $\psi \mid L\colon L \to R^{d+k}$ is totally geodesic, which means, in this case, that the image $\psi(L)$ is a portion of an n-dimensional plane in R^{d+k}.

For an adapted frame field e, the restrictions of the forms ϕ_{ra} to a leaf L of $\mathscr{N}$ are the Codazzi forms (relative to e) for L as a submanifold of M. Thus we must show $\phi_{ra}(e_b) = 0$. The Codazzi equation for $\tau_{a\alpha}$ is

$$d\tau_{a\alpha} = -\sum \phi_{ai} \wedge \tau_{i\alpha} - \sum \tau_{a\beta} \wedge \theta_{\beta\alpha}\,.$$

By (1) this reduces to

$$\sum \phi_{ar} \wedge \tau_{r\alpha} = 0\,. \tag{3}$$

By an earlier remark we can assume that e has been chosen so that at an arbitrary point m the operator T_{e_d} is one-to-one on $\mathscr{N}^{\perp}(m)$. Applying the 2-form of (3) to the vectors e_b, e_d at m, we get $\Sigma\phi_{ar}(e_b)\,\tau_{r\alpha}(e_d) = 0$. By the one-to-one property of T_{e_d}, the $(d - n) \times k$ matrix $(\tau_{r\alpha}(e_d))$ has rank $d - n$, and it follows that $\phi_{ar}(e_b) = 0$. It remains to prove

LEMMA 1. *The leaves of* $\mathscr{N}$ *are complete.*

Let γ: $[0, c) \rightarrow L$ be a geodesic ray in a leaf L of $\mathscr{N}$. It suffices to show that γ can be extended, as a geodesic of L, over the half-line $[0, \infty)$. Suppose this cannot be done; that is, suppose γ as given is maximal. Since M is complete, γ can be extended as a geodesic $\tilde{\gamma}$ of M. Now, since L is totally geodesic in M, it follows that $\tilde{\gamma}(c)$ is not in G. (If it were, $\tilde{\gamma}$ would provide the required extension.) Again using the facts that L is totally geodesic in M (flat) and that $T_{\gamma'} = 0$, we can choose the frame field e so that γ is an integral curve of e_1 and e is Euclidean parallel on γ. Furthermore, we can arrange for T_{e_d} to have rank $d - n$ on $M_{\gamma(0)}$. Note that e, and thus the forms associated with it, are defined only inside the set G; however e_d can be extended by parallel translation along $\tilde{\gamma}$ to the point $p = \tilde{\gamma}(c)$. Let T be the operator T_{e_d} at p. Since p is not in G, $\mathscr{N}(p)$ has dimension $\nu(p) > n$. But T is zero on $\mathscr{N}(p)$; hence rank $T \,|\, M_p < d - n$. Our aim now is to show the impossibility of this drop in rank of T_{e_d} along $\tilde{\gamma}$. The contradiction will prove the lemma and thereby Theorem 1. To obtain it, we need some further lemmas.

LEMMA 2. *The covariant derivative* $e_1(T_{e_d}(e_r))$ *of the vector field* $T_{e_d}(e_r)$ *on* γ *is* $-\Sigma_s\, \phi_{s1}(e_r)\, T_{e_d}(e_s)$.

Proof. Since e is Euclidean parallel on γ,

$$e_1(T_{e_d}(e_r)) = \sum_{\alpha} e_1(\tau_{\alpha r}(e_d))\, e_\alpha\,.$$

Now apply (2) to e_1, e_d. The parallelism of e implies that the forms ϕ_{ij}, $\tau_{i\alpha}$, $\theta_{\alpha\beta}$ are all zero on $\gamma' = e_1$; hence we get $e_1(\tau_{r\alpha}(e_d)) = \tau_{r\alpha}([e_1, e_d])$. The first structural equation applied to e_1, e_d yields

$$[e_1, e_d] = -\sum_i \phi_{i1}(e_d)\, e_i\,.$$

Hence

$$e_1(\tau_{\alpha r}(e_d)) = -\sum_s \phi_{s1}(e_d)\, \tau_{\alpha r}(e_s)\,.$$

Now the left side of this equation is symmetric in r and d; reversal of r and d on the right side gives the coordinate form of the required result.

LEMMA 3. *On* γ, $\int_0^t \Sigma \phi_{r1}(e_r) \to +\infty$ *as* $t \to c$.

Proof. Let W be the function whose value at $t \in [0, c]$ is the multivector $T_{e_d}(e_{n+1}) \wedge \cdots \wedge T_{e_d}(e_d)$. Using the previous lemma, we find that for $t < c$, the covariant derivative of this function is $e_1(W) = -(\Sigma\phi_{r1}(e_r))W$. Hence, for $t < c$,

$$W(t) = \left\{\exp\left[-\int_0^t \Sigma\, \phi_{r1}(e_r)\right]\right\} W_t(0),$$

where $W_t(0)$ is the result of the parallel translation of W(0) along γ to $\gamma(t)$. But from before we know that T_{e_d} has rank strictly less than $d - n$ on M_p $(p = \tilde{\gamma}(c))$. Hence $W(c) = 0$, and the result follows.

LEMMA 4. *On* γ, $e_1(\phi_{r1}(e_s)) = -\Sigma_q\, \phi_{r1}(e_q)\, \phi_{q1}(e_s)$.

Proof. Applying the second structural equation for the form ϕ_{r1} to the vectors e_1 and e_s along γ, and using $\phi_{ij}(e_1) = 0$, we get

$$e_1(\phi_{r1}(e_s)) = \phi_{r1}([e_1, e_s]) = -\sum_i \phi_{i1}(e_s)\, \phi_{r1}(e_i).$$

Since $\phi_{ra}(e_b) = 0$, the index i may here be replaced by q $(n + 1 \leq q \leq d)$.

We are now in a position to complete the proof of Lemma 1. If $t \in [0, c)$, let $P_{rs}(t)$ be the value of $\phi_{r1}(e_s)$ at $\gamma(t)$. Then $P = (P_{rs})$ is a differentiable $(d - n) \times (d - n)$ matrix-valued function on $[0, c)$. Lemmas 3 and 4 may be written in the forms

(L3) $\int_0^t \text{trace } P \to +\infty$ as $t \to c$, and

(L4) $P' = -P^2$.

The differential equation L4 has the solution

$$P(t) = P(0)\,(I + t\,P(0))^{-1} \quad \text{for } t \in [0, c). \tag{4}$$

We show by induction on $d - n \geq 1$ that L3 and L4 are contradictory; this will complete the proof of Theorem I. The contradiction is obvious when $d - n = 1$, since the relation $\int_0^t P \to +\infty$ as $t \to c$ is incompatible with $P' \leq 0$. Suppose the contradiction holds for $d - n < h$, where $h > 1$. First consider the case where the $h \times h$ matrix P(0) is singular. We can assume that the first column of P(0) is 0. By (4) the same is true of all P(t). Then the matrix function $P = (P_{rs})$ $(n + 2 \leq r, s \leq d)$ still satisfies L3 and L4, and therefore we have a contradiction. Now suppose P(0) is nonsingular. From L4 it follows that the determinant Δ of P satisfies the differential equation $\Delta' = -(\text{trace } P)\Delta$. Thus L3 implies that $\Delta \to 0$ as $t \to c$. From (4) we get

$$\Delta(t) = \Delta(0)\{\det(I + t\,P(0))\}^{-1}.$$

But $\Delta(0) \neq 0$ and $\det(I + t\,P(0))$ is bounded on $[0, c)$; hence Δ can *not* approach 0 as $t \to c$.

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3. CYLINDRICAL IMMERSIONS

We say that an isometric immersion $\psi: M^d \to R^{d+k}$ is n-*cylindrical* provided M and ψ can be expressed as Riemannian products $M^d = B^{d-n} \times R^n$ and $\psi = \overline{\psi} \times 1$, where $\overline{\psi}$ is an isometric immersion of B^{d-n} in R^{d+k-n} and 1 is the identity map of R^n.

THEOREM 2. *Let* M^d *be a complete, flat Riemannian manifold. An isometric immersion* $\psi: M^d \to R^{d+k}$ *is* n-*cylindrical if*

(a) *the relative nullity function* ν *has constant value* n, *and*

(b) *the relative curvature of* ψ *is zero.*

We explain the second condition: let N be the bundle of normal k-frames of M relative to ψ; that is, let

$$N = \{(m, f) \mid m \in M, \text{ and } f \text{ is a } k\text{-frame of } R^{d+k} \text{ orthogonal to } d\psi(M_m)\}$$

with natural bundle-structure. The Euclidean connection of R^{d+k} induces a natural connection on N. It is the curvature form of this connection which we assume to be zero. In terms of the Codazzi forms $\tau_{i\alpha}$ of an adapted frame field e, this means $\Sigma\, \tau_{\alpha i} \wedge \tau_{i\beta} = 0$. In invariant terms, it says that the restriction of the operator $[T_x, T_y]$ to $(d\psi(M_m))^{\perp}$ is zero for all x, y $\in M_m$. (This is automatically true if x or y is in $\mathcal{N}(m)$.) Flatness of M is equivalent to $[T_x, T_y] \mid d\psi(M_m) = 0$. Thus, under condition (b), *any two difference operators* T_x *and* T_y *are commutative on* $(R^{d+k})_{\psi(m)}$.

Conditions (a) and (b) above are not necessary for an isometric immersion to be n-cylindrical. In the theorem of Hartman and Nirenberg (referred to in the Introduction) we have k = 1; hence (b) holds automatically and (a) can be dispensed with by the use of the special fact that disjoint d-planes in R^{d+1} are parallel.

Proof of Theorem 2. Condition (a) implies that $\mathcal{N}$ is a differentiable field of n-planes on all of M. We know that ψ carries the leaves L of $\mathcal{N}$ isometrically onto n-planes in R^{d+k}. In the proof that ψ is n-cylindrical, the main point is to show that all these planes $\psi(L)$ are parallel in R^{d+k}. The relative position of the leaves in M can be measured as follows: fix an adapted frame field e on a neighborhood of m $\in$ M, and let P_{e_a} be the linear operator on $\mathcal{N}^{\perp}(m)$ such that

$$P_{e_a}(e_s) = \sum_r \phi_{ra}(e_s)\, e_r .$$

Extending linearly, we get for each x $\in \mathcal{N}(m)$ a linear operator P_x on $\mathcal{N}^{\perp}(m)$. These operators are related to the difference operators by

LEMMA 5. *If* $x \in \mathcal{N}(m)$ *and* $y \in \mathcal{N}^{\perp}(m)$, *then* $T_{P_x(y)} = T_y \circ P_x$ *on* $\mathcal{N}^{\perp}(m)$.

Proof. We have

$$T_{e_r}(P_{e_a}(e_s)) = \sum_{\alpha, q} \phi_{qa}(e_s)\, \tau_{\alpha q}(e_r)\, e_\alpha .$$

Equation (3) shows we can here interchange s and r, so that this vector equals $T_{e_s}(P_{e_a}(e_r))$. Hence for x, y as above and $z \in \mathcal{N}^\perp(m)$, we get

$$T_y(P_x(z)) = T_z(P_x(y)) = T_{P_x(y)}(z).$$

LEMMA 6. *Each operator* P_x *is symmetric.*

Proof. Let $x \in \mathcal{N}(m)$, and choose $y \in \mathcal{N}^\perp(m)$ so that T_y is one-to-one on $\mathcal{N}^\perp(m)$. Let

$$A = d\psi(\mathcal{N}(m)) + T_y(d\psi(\mathcal{N}(m))) \subset (R^{d+k})_{\psi(m)}.$$

One can verify that the subspace A is invariant under both T_y and $T_{P_x(y)}$. Furthermore, the restriction $T_y|A$ is non-singular. Since ψ has relative curvature zero, the operators T_y and $T_{P_x(y)}$, hence also $(T_y|A)^{-1}$ and $T_{P_x(y)}|A$, commute—and are skew-symmetric. Thus $(T_y|A)^{-1} \circ (T_{P_x(y)}|A)$ is a symmetric operator which, by the preceding lemma, agrees with P_x on $\mathcal{N}^\perp(m)$.

Note that this lemma implies that $\mathcal{N}^\perp$ is integrable. In fact, from the first structural equation, we get

$$[e_r, e_s] = \sum_i (\phi_{ri}(e_s) - \phi_{si}(e_r))\, e_i.$$

So, since the matrix of P_{e_a} is symmetric, we get $[e_r, e_s] \in \mathcal{N}^\perp$, which implies integrability. For $x \in \mathcal{N}(m)$, P_x is actually a second fundamental form operator of the leaf K(m) of $\mathcal{N}^\perp$ through m and is thus independent of the choice of frame field used in its definition.

LEMMA 7. $P_x = 0$ *for all* $x \in \mathcal{N}(m)$, $m \in M$.

Proof. If $x \in \mathcal{N}(m)$, let γ be the geodesic of L = L(m) with initial velocity x. We know that γ can be defined on the whole real line, and as before we can assume that e is Euclidean parallel along γ and that there $e_1 = \gamma'$. The matrix P(t) of P_{e_1} at $\gamma(t)$ is the matrix used in the proof of Lemma 1, hence it obeys the differential equation $P' = -P^2$. Since P_x is symmetric, we can assume P(0) is diagonal; but by (4) this implies that every P(t) is diagonal. Thus the differential equation reduces to $P'_{rr} = -(P_{rr})^2$. Since this holds on the entire real line, $P_{rr} = 0$, hence $P_x = 0$.

This lemma, together with the earlier result $\phi_{ra}(e_b) = 0$, implies $\phi_{ra} = 0$. It follows that both $\mathcal{N}$ and $\mathcal{N}^\perp$ are parallel on M. Since all difference operators are zero on $\mathcal{N}(m)$, we conclude that ψ carries the leaves of $\mathcal{N}$ to *parallel* n-planes in R^{d+k}.

Fix a point $m_0 \in M$, and suppose ψ carries m_0 to the origin of R^{d+k}. Let K_0 and L_0 be the leaves of $\mathcal{N}$ and $\mathcal{N}^\perp$ through m_0. Then let R^n be the vector subspace $\psi(L_0)$ of R^{d+k}, with R^{d+k-n} the orthogonal vector subspace.

LEMMA 8. *If* L *is a leaf of* $\mathcal{N}$, *and* K *is a leaf of* $\mathcal{N}^\perp$, *then* $K \cap L$ *contains exactly one point.*

Proof. Since all leaves of $\mathcal{N}$ are carried to n-planes parallel to $\psi(L)$, it follows that $\psi(K)$ is contained in some (d + k - n)-plane orthogonal to $\psi(L)$. Thus if $p, q \in K \cap L$, then $\psi(p) = \psi(q)$. But ψ is one-to-one on L, hence p = q. To show that $K \cap L$ is non-empty, let $\pi: R^d \to M$ be the simply-connected Riemannian covering of M, and let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the plane-fields corresponding under $d\pi$ to

$\mathscr{N}$ and $\mathscr{N}^{\perp}$. Since $\mathscr{P}$ and $\mathscr{P}^{\perp}$ are parallel, the deRham decomposition theorem applies; in particular, each leaf of $\mathscr{P}$ meets each leaf of $\mathscr{P}^{\perp}$. These leaves are mapped onto corresponding leaves below, by π. Hence the result follows.

We can easily deduce from this lemma that $\mathscr{N}$ and $\mathscr{N}^{\perp}$ give a product structure to M. In fact, the function $\mu: M \to K_0 \times L_0$ that sends m to

$$(m_1, m_2) = (L(m) \cap K_0, K(m) \cap L_0)$$

is an isometry.

The proof of Theorem 2 will now be completed by showing that

$$\psi = (\psi \mid K_0 \times \psi \mid L_0) \circ \mu .$$

With the notation above, this may be rewritten as $\psi(m) = \psi(m_1) + \psi(m_2)$, for all $m \in M$. If $m \in M$, let $\sigma: [0, 1] \to K_0$ be a curve from the fixed point m_0 to m_1. Because of the product structure on M, there exists a parallel vector field X of M, defined on σ, such that $\exp(X(0)) = m_2$, $\exp(X(1)) = m$. (Here exp is the map whose value on a tangent vector x is the point attained in unit time by the geodesic with initial velocity x.) The image $d\psi(X)$ of X is Euclidean parallel, since each vector of the field X is contained in a plane of $\mathscr{N}$. If $x \in (R^{d+k})_p$, let $\langle x \rangle$ be the canonically corresponding element of R^{d+k}, that is, let $\langle x \rangle = q - p$, where $q = \exp(x)$. Using the facts above, we get

$$\psi(m_2) = \psi(\exp(X(0))) = \exp(d\psi(X(0))) = \langle d\psi(X(0)) \rangle = \langle d\psi(X(1)) \rangle$$
$$= \psi(\exp(X(1))) - \psi(\sigma(1)) = \psi(m) - \psi(m_1) .$$

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