# ON THE ALGEBRAIC CLOSURE OF A PLANE SET

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1. Let Z be a set of complex numbers. Where convenient, Z will be identified in the natural way with the corresponding subset of the Euclidean plane E. We shall always assume that $\{-1, 0, 1\} \subset Z$; otherwise, Z is arbitrary. Define $Z_1 = Z$,

$$Z_{n+1} = \left\{ z: \sum_{j=0}^{N} a_j z^j = 0,\ \{a_0, \cdots, a_N\} \subset Z_N, N \geq 1,\ a_N \neq 0 \right\} \quad (n = 1, 2, \cdots).$$

Then $Z_1, Z_2, \cdots$ is an ascending sequence of sets, and we call the set

$$Z_\omega = \lim_{n \to \infty} Z_n = \bigcup_{n=1}^{\infty} Z_n$$

the *algebraic closure* of Z. It is clearly the smallest set that contains Z and is algebraically closed in the usual sense. In this note we shall study some properties of the sets $Z_\omega$ and of the relation of $Z_\omega$ to Z. Thus, we consider, in a sense, the algebraic closure apart from any algebraic structure.

If A denotes a set, an A-*equation* is an algebraic equation in a single unknown and with all coefficients in A. Any topological terms refer to the usual topology of E, and any group-theoretic ones to the ordinary multiplication of complex numbers.

2. LEMMA 1. $Z_\omega - \{0\}$ *is an Abelian group containing for every positive integer* n *every* n-*th root of every one of its elements.*

This follows directly from the definition of $Z_\omega$ and from the observation that if $z_1, z_2 \in Z_\omega$, then the equations $z_1 z - 1 = 0$, $z + z_2 = 0$, $z/z_1 - z_2 = 0$, and $z^n - z_1 = 0$ are $Z_\omega$-equations.

THEOREM 1. $Z_\omega$ *is dense in* E.

By Lemma 1, $Z_\omega$ contains the group U of all roots of unity. It is easily verified that $Z_\omega$ contains two positive numbers, a and b, such that log a/log b is irrational. For instance, $z^2 - z - 1 = 0$ is a $Z_\omega$-equation, so that $a = (1 + 5^{1/2})/2 \in Z_\omega$; also, $z^2 - z - a = 0$ is a $Z_\omega$-equation, and therefore $b = [1 + (3 + 2 \cdot 5^{1/2})^{1/2}]/2 \in Z_\omega$. Suppose now that log a/log b is rational. Then $a^p = b^q$ for some positive integers p and q, which implies that $(3 + 2 \cdot 5^{1/2})^{1/2} = r + 5^{1/2} s$, with r and s rational. This implies further that $r^2 - 3rs + 5s^2 = 0$, which leads to $r = s = 0$, and this is a contradiction.

It follows that the module $\{n \log a + m \log b\}$, where n and m range independently over the rational integers, contains numbers of arbitrarily small absolute value, and hence is dense on the real line. Therefore the group S, generated by a

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and b, is dense in the positive reals. Finally, $SU = \{z: z = su, s \in S, u \in U\}$ is dense in E, and so is $Z_\omega$, since it contains SU.

LEMMA 2. *If* $Z_\omega$ *contains an open set, then* $Z_\omega = E$.

If $Z_\omega$ contains an open set, it must contain an open disk

$$D = \{z: |z - z_1| < r, \; r > 0, \; z_1 \neq 0\}.$$

Let $z_0 \in Z_\omega$ be arbitrary, $z_0 \neq 0$. As u varies over D, the roots of the $Z_\omega$-equation $z_1 z - z_0 u = 0$ fill out the disk $z_0 D/z_1$. It follows now from Theorem 1 that $\{z: |z| \geq 1\} \subset Z_\omega$; hence, by Lemma 1 and the assumption $0 \in Z$, it follows that $Z_\omega = E$.

For the next few lemmas we require the concept of vector addition of plane sets: if X and Y are sets of complex numbers, then $X + Y = \{z: z = x + y, x \in X, y \in Y\}$.

LEMMA 3. *If* X *is a Jordan curve, then the set* $2X = X + X$ *contains a translate of the interior of* X.

Let $z_0 \in X$ be arbitrary, and let z be any point inside X. Given a point $u \in X$, let v be the fourth vertex of the parallelogram $z_0 uzv$, the four points being the vertices in cyclic order. As u ranges over X, the corresponding $v = v(u)$ traces out a curve T which is congruent to X, and passes through the interior of X. Hence, by the Jordan separation property, T cuts X, which means that there exist points $u_0$ and $v_0$ on X, such that $z_0 u_0 z v_0$ is a parallelogram. Therefore $z = u_0 + v_0 - z_0$, which proves the lemma.

LEMMA 4. *If* X *is an arc other than a straight segment, then* 2X *contains a Jordan curve.*

Without loss of generality, let X be a simple arc with the end-points 0 and a. For $p, q \in X$ $(p \neq q)$, let $X(p, q)$ denote the subarc of X with the end-points p and q. For any $p \in X$ the translate $X + \{p\}$ will be denoted by $X_p$. Since X is not a segment, it follows that there exists a translate $X_p$ such that $X(p, a)$ is not a subset of $X_p$. Let t be any point of $X(p, a)$ which is not in $X_p$, and let

$$W = X_p \cup X(p, t) \cup (X(p, t) + \{a\}) \cup X_t.$$

W is then a closed circuit consisting of four arcs, each of which is a subset of 2X. By its construction, W does not reduce to an arc (although it may contain a subarc consisting of multiple points of W). Hence W contains a Jordan curve, and the lemma is proved.

THEOREM 2. *If* $Z_\omega$ *contains an arc* J, *then* $Z_\omega = E$.

There are two cases to consider, depending on whether J is or is not a subarc of a logarithmic spiral with the origin as its pole (degenerate cases of circles with their center at the origin, and straight lines through the origin, are included with such spirals). Suppose first that J is such a subarc. Let $y \in Z_\omega$ be a point, to be chosen later, and put

$$J_y = \{w: w = yz, \; z \in J\} \subset Z_\omega.$$

We have then

$$K = \{z: z^2 + az + b = 0, a \in J_y, b \in J_y\} \subset Z_\omega,$$

and since $Z_\omega$ is dense in E, it is easily proved that for some $y \in Z_\omega$ the set

$$\{w: w = a^2 - 4b, \ a \in J_y, \ b \in J_y\}$$

contains an open set. Hence K contains an open set, and so, by Lemma 2, $Z_\omega = E$.

Suppose next that J is not a subarc of a spiral. Without loss of generality, assume that J does not contain the origin, and that it subtends at the origin an angle less than $\pi/4$. Let $\phi$: E → E be the mapping which sends the point $z = r \exp i\theta$ into the point $w = \log r + i\theta$ (under our conditions on J, no difficulties arise from the multiplicity of the argument or from the singularity of the mapping at the origin). Now $L = \phi(J)$ is an arc other than a straight segment, and we have

$$J^4 = \{z: z = x_1 x_2 x_3 x_4, \ x_i \in J, \ i = 1, 2, 3, 4\} = \phi^{-1}(2L + 2L).$$

By Lemmas 3 and 4, 2L + 2L contains an open set, hence so does $J^4$. By Lemma 1, $J^4 \subset Z_\omega$, and the theorem follows from Lemma 2. The following is an easy consequence of Theorem 2:

THEOREM 3. *If* $Z_\omega \neq E$, *then every point* $z \in E$ *is a condensation point of* $E - Z_\omega$.

Suppose that this is false, and let $z \in E$ be a point which is not a condensation point of $E - Z_\omega$. Then there exists an open disk D about z, such that $D \cap (E - Z_\omega)$ is at most countable. Hence $Z_\omega$ contains an arc (for instance, an open segment) lying inside D. Therefore $Z_\omega = E$, by Theorem 2, which is a contradiction.

3. In this section we show that there exist sets Z which are "thin" or "small" in a certain fairly strong sense, but for which $Z_\omega = E$ nevertheless. We remark first that if $Z_\omega = E$, then Z must be uncountable. Further, by a slight modification of the procedure used to prove the existence of Hamel bases, it may be proved that there exist uncountable sets Z for which $Z_\omega \neq E$.

Let X be a plane set, and let $\delta(X)$ be its diameter. For any numbers p and $\sigma$ ($p \geq 0$, $\sigma > 0$) let

$$M_p(X, \sigma) = \inf \sum_{j=1}^{\infty} [\delta(A_j)]^p,$$

where the infimum is taken over all coverings $\bigcup_{j=1}^{\infty} A_j$ of X subject to the condition $\delta(A_j) < \sigma$ ($j = 1, 2, \cdots$). The limit

$$M_p(X) = \lim_{\sigma \to 0} M_p(X, \sigma)$$

exists, since $M_p(X, \sigma)$ is a nonincreasing function of $\sigma$, and it is called the *p-dimensional outer Hausdorff measure* of X. If $0 < M_p(X) < \infty$, then $M_p(X) = 0$ for $q < p$, and $M_q(X) = \infty$ for $q < p$. The *Hausdorff dimension* d(X) is defined to be the infimum of the numbers p for which $M_p(X) = 0$.

THEOREM 4. *There exists a linear set* Z *such that* $d(Z) = 0$ *and* $Z_\omega = E$.

A complex number x is called a *Liouville number* if the inequality

$$0 < |x - p/q| < q^{-m}$$

has a solution, for each positive m, in rational integers p and q $(q > 1)$. It follows that a Liouville number is real, and it is well known that it must be transcendental. Let L, R, and P denote the sets of all Liouville numbers, all rational numbers, and all rational integers, respectively. Let $\{r_i\}$ $(i = 1, 2, \cdots)$ be an enumeration of the set $R - P$, and let $r_i = p_i/q_i$, where $p_i, q_i \in P$ and $q_i > 1$ $(i = 1, 2, \cdots)$. We have then

$$L \subset \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (p_i/q_i - q_i^{-n}, p_i/q_i + q_i^{-n}),$$

from which it follows easily that $d(L) = 0$. Let $Z = L \cup P$; then $d(Z) = 0$. Now every real number other than 0 is the root of a linear equation whose coefficients are Liouville numbers (see [1]). Hence, by the definition of $Z_2, Z_3, \cdots$, the set $Z_2$ contains the real axis, hence $Z_3 = E$, and, *a fortiori*, $Z_\omega = E$.

4. We conclude with some problems. What is the structure of the sets $Z_1, Z_2, \cdots, Z_\omega$, when $Z = \{-1, 0, 1\}$? Is 2 contained in any of these sets? Is it true that if Z is measurable with respect to the plane Lebesgue measure, then either $Z_\omega = E$ or else, for each measurable set A, $\mu(A) = \mu(A - Z_\omega)$? Is it true that each measurable subgroup of E is a subset of a union of logarithmic spirals, each one of which intersects the subgroup in a set dense on that spiral? Let Z contain -1, 0, and 1, and call $z \in Z$ $(z \neq z^2)$ *superfluous* if $(Z - \{z\})_\omega = Z_\omega$; let also the *core* of Z be the set of its non-superfluous elements z $(z \neq z^2)$; what is the structure of the cores of sets, and of sets with empty cores? Let $\{-1, 0, 1\} \subset Z$, and define $Z^*$ to be the smallest group containing Z, and such that with every $z \in Z^*$ and every positive integer n, $Z^*$ contains also every n-th root of z; which propositions proved here for $Z_\omega$ hold also for $Z^*$? What is the structure of the quotient groups $Z_\omega/Z^*$?

## REFERENCE

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