TYPES OF AMBIGUOUS BEHAVIOR OF ANALYTIC FUNCTIONS

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Let $w = f(z)$ be a complex-valued function, defined on the open disk D composed of all complex numbers z such that $|z| < 1$, and let z_0 be a point on the unit circle C, that is, on the boundary of D. We recall three definitions:

(1) The *cluster set* of f at z_0, $C(f, z_0)$, is the set of all points w on the Riemann sphere such that for each open set U containing z_0, every open set containing w meets the set $f(U \cap D)$. That is, w is in $C(f, z_0)$ provided there exists a sequence $\{z_j\}$ in D such that $z_j \to z_0$ and $f(z_j) \to w$.

(2) The *boundary cluster set* of f at z_0, $C_B(f, z_0)$, is the set of all points w such that for each open set U containing z_0, every open set containing w meets $\bigcup C(f, z')$, where the union is taken over all z' in $U \cap (C - z_0)$. If f is continuous, $C(f, z_0)$ is connected, but $C_B(f, z_0)$ need not be; however, if $C_B(f, z_0)$ is not connected, it has exactly two components, and these coincide with the right and left boundary cluster sets (definition obvious!), respectively. A semi-classical theorem of Iversen asserts that if f is meromorphic, then the boundary of $C(f, z_0)$ is contained in $C_B(f, z_0)$. (See, for example, [6, Theorem 1'].)

(3) For an *arc* A *terminating at* z_0 on C (we shall always mean, by this expression, an arc lying in D except for one endpoint at z_0), the *arc-cluster set* of f on A, $C(f, A, z_0)$, is the set of all points w such that for every open set U containing z_0, every open set containing w meets $f(A \cap U)$. Each arc-cluster set is connected, if f is continuous.

It may happen, even with bounded analytic functions, that at a point z_0 in C there exist two arcs, A_1 and A_2, terminating at z_0, for which the sets $C(f, A_1, z_0)$ and $C(f, A_2, z_0)$ are disjoint. If this does occur at z_0, we shall say that z_0 is a *point of disjoint cluster sets* [10]. Bagemihl has shown [1] that even for a purely arbitrary function the set of points of disjoint cluster sets is at most countable. The restriction of the discussion to analytic functions does not strengthen the conclusion. In fact, as Gross [7, Section 8] has shown, corresponding to each countable set $X = \{x_n\}$ on C, we can choose positive numbers $\{a_n\}$ such that $\{x_n\}$ is precisely the set of points of disjoint cluster sets of the bounded function

$$\exp\left(\sum_{n=1}^{\infty} a_n \cdot \frac{z_n + x_n}{z_n - x_n}\right).$$

If the condition of boundedness is replaced by the weaker condition that f be of bounded characteristic, it is even possible (see [3] and [9]) to require that for every x_n in X, the disjoint sets $C(f, A_1, x_n)$ and $C(f, A_2, x_n)$ each consist of one point (a point x_n exhibiting this phenomenon is called an *ambiguous point*).

Lohwater pointed out in [8] that the property of being a point of disjoint cluster sets is a special case of a more general property, and he proposed the investigation

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of that property. We shall say that a point z_0 on C has *property* P_n, for some integer $n \geq 2$, if there exist in $D \cup z_0$ n arcs, $A_1, A_2, \cdots, A_n$, all ending at z_0, such that for each integer k $(1 \leq k \leq n)$ there is at least one point that belongs to all the cluster sets $C(f, A_j, z_0)$ $(j \neq k)$, but such that no point belongs to all the cluster sets $C(f, A_j, z_0)$ $(j = 1, 2, \cdots, n)$. A point of disjoint cluster sets has property P_2. Unfortunately, for $n > 2$ there is no analogue of Bagemihl's theorem. Bagemihl, Piranian, and Young [2] have given an example of a bounded analytic function in D such that the set of points with property P_3 is a Cantor set on C, and an example of an unbounded analytic function in D such that each point of C has property P_3.

It is rather easy to give examples of analytic functions, even schlicht functions, defined in D such that at some point z_0 in C each pair of arc cluster sets intersects, but such that $C(f, z_0)$ is not a point. For example, let f be a conformal map of D onto the open set consisting of all points z lying above the closure of the graph of $y = \sin(1/x)$, and let z_0 be the point on C which corresponds to the prime end whose impression is the interval $[-i, i]$ of the imaginary axis. Here the discontinuity of f is not really bad; the more interesting discontinuities occur at points z_0 where $C(f, z_0) - C_B(f, z_0)$ is not empty. Our principal theorem is concerned with conditions under which such points have property P_n for $n \geq 2$. On the other hand, we show by an example that the nonemptiness of $C(f, z_0) - C_B(f, z_0)$ is not sufficient, even for bounded analytic functions, to guarantee property P_n for any $n \geq 2$.

THEOREM 1. *Let* f(z) *be meromorphic in the open unit disk* D, *and let* z_0 *be a point of the unit circle* C *such that the set* $C_B(f, z_0)$ *is not connected. Then the point* z_0 *has property* P_2. *If in addition* f *is bounded, then the point* z_0 *has property* P_n *for all* $n \geq 2$.

Proof. Since $C(f, z_0)$ is connected, but $C_B(f, z_0)$ is not, the set

$$C(f, z_0) - C_B(f, z_0)$$

is not empty; call it E. By the theorem of Iversen mentioned in the first paragraph, the set E is open. The set $C_B(f, z_0)$ is the union of two disjoint continua M_1 and M_2, and at least one component U of E has boundary points both in M_1 and M_2. By the local connectivity of the Riemann sphere, the boundary of U is contained in the boundary of E, and it follows that the boundary of U consists of two continua, N_1 and N_2, where N_j is a subset of M_j for $j = 1, 2$.

Let $R(f, z_0)$ denote the *range of* f *at* z_0, that is, the set of all points w such that every neighborhood of z_0 contains a point z for which $w = f(z)$. The Gross-Iversen Theorem [6, Theorem 2] states that $R(f, z_0)$ contains all but at most two points of U. Let J be a simple closed curve in $U \cap R(f, z_0)$ that separates N_1 from N_2 and, for topological simplicity, that does not pass through the image of any point z for which $f'(z) = 0$. Then each component of $f^{-1}(J)$ is either a simple closed curve, or else it is homeomorphic to an open interval, since f acting on $f^{-1}(J)$ is a local homeomorphism. (The restriction that $f'(z) \neq 0$ on the preimage of J presents no difficulty; for there exist only countably many points w on the Riemann sphere such that f' vanishes at some preimage of w, and the curve J can therefore always be constructed as an appropriate polygon.)

Let V denote the closure of a disk with center at z_0, and let $v = V \cap C - z_0$. We suppose that V is chosen small enough so that the set $\bigcup_{z \in v} C(f, z)$ lies at a positive distance from the set $f^{-1}(J)$. Then only a finite number of components of $f^{-1}(J)$ meet Bdry V. Otherwise some point w of $\overline{D} \cap$ Bdry V would be the limit of a sequence of points from distinct components of $f^{-1}(J)$. The point w cannot belong to C, for then C(f, w) would meet J. But w cannot belong to D, since on $f^{-1}(J)$ the map f is a local homeomorphism. Also, z_0 is the only point of $C \cap V$ that can possibly be a limit point of a component of $f^{-1}(J)$. If there is a component K of $f^{-1}(J)$ that has z_0 as limit point, then the set $K \cup z_0$ contains an arc A terminating at z_0, and $C(f, A, z_0)$ is contained in J. In fact, $C(f, A, z_0)$ consists either of a point or else of all of J.

Next we show that some component of $f^{-1}(J)$ has z_0 as a limit point. If no component of $f^{-1}(J)$ has z_0 as limit point, then we can join each pair of points x and x' on $V \cap C$ by an arc α in $V \cap D \cup (x \cap x')$ that does not meet the set $f^{-1}(J)$. The connected set $f(\alpha)$ meets each of $C(f, x)$ and $C(f, x')$, but it does not meet J. Therefore, for x fixed, each set $C(f, x')$ lies in the same complementary domain of J as does $C(f, x)$. It follows that $C_B(f, z_0)$ lies in that complementary domain, which contradicts the fact that J separates N_1 from N_2. Therefore, one of the components of $f^{-1}(J) \cap V$ contains an arc terminating at z_0.

Since there are uncountably many disjoint simple closed curves satisfying the conditions imposed on J, there are uncountably many arcs approaching z_0 such that the arc cluster sets on any two are disjoint. Thus z_0 has property P_2 in a strong form.

Up to this point, we have assumed only that f is meromorphic. We now show that if f is also bounded, it has property P_n for $n > 2$. Note first that there is at most one asymptotic value of f at z_0, by Lindelöf's Theorem. Let $J_1, J_2, \cdots, J_n$ be a set of n simple closed curves such that (1) each curve J_k lies in $R(f, z_0)$ and separates N_1 from N_2; (2) no curve J_k passes through the image of a point z for which $f'(z) = 0$; (3) no curve J_k passes through an asymptotic value of f at z_0; (4) each n - 1 of the curves have a point in common; and (5) no point lies on all the curves. From the preceding paragraphs it follows that there are n arcs $A_1, A_2, \cdots, A_n$ in $D \cup z_0$ such that $C(f, A_k, z_0) = J_k$ $(k = 1, 2, \cdots, n)$. Thus z_0 has property P_n for each $n \geq 2$.

Now let $C_{BR}(f, z_0)$ and $C_{BL}(f, z_0)$ denote respectively the right and left boundary cluster sets at z_0. We have the following corollary to Theorem 1.

THEOREM 2. *Let* f *be meromorphic in the open unit disk* D, *and let* z_0 *be a point of the unit circle such that there exists an arc* A *terminating at* z_0 *for which* $C(f, A, z_0) \cap C_{BR}(f, z_0)$ *is empty. Then the point* z_0 *has property* P_2. *If in addition* f *is bounded, then the point* z_0 *has property* P_n *for all* $n > 2$.

Proof. Join the endpoint of A in D to C by an arc A' not terminating at z_0. Let D' denote the component of D - A - A' whose boundary contains an arc of C abutting on z_0 from the right. There is a conformal homeomorphism ϕ of D onto D', which can be extended to a continuous homeomorphism of Cl D onto Cl D'. Let z_1 denote $\phi^{-1}(z_0)$, and consider the map $f\phi$ defined in D. We have

$$C_{BR}(f\phi, z) = C_{BR}(f, z_0) \quad \text{and} \quad C_{BL}(f\phi, z_0) = C(f, A, z_0).$$

Thus we can apply Theorem 1 to the map $f\phi$ and assert that z_1 has property P_2, and if f is bounded, that z_1 has property P_n for $n \geq 2$. However, the fact that ϕ is a homeomorphism implies that z_0 has property P_n for f whenever z_1 has that property for $f\phi$.

A special case of Theorem 2 is the following result.

THEOREM 3. *Let* f(z) *be meromorphic in* D, *and suppose that there exists an open arc* A *of* C *on which the modulus* $|f(re^{i\theta})|$ *has radial limit* 1 *for almost all points* $e^{i\theta}$ *in* A. *Suppose that the point* z_0 *lies on the open arc* A *and is not the* *limit of zeros or poles of* f(z). *Suppose further that* $C(f, z_0) - C_B(f, z_0)$ *is not empty. Then* z_0 *has property* P_2. *If* f *is bounded,* z_0 *has property* P_n *for* $n \geq 2$.

Proof. A theorem of Carathéodory's [4] implies that at each singular point of f in A the cluster set is either the set $M = C \cup D$ or the set N that is the complement of D, or the entire plane. If z_0 is an isolated singularity of f, the set $C_B(f, z_0)$ is C. If it is not an isolated singularity, the set $C_B(f, z_0)$ is either M or N; it cannot be $M \cup N$, since it is a proper subset of $C(f, z_0)$. Therefore there exists a subarc A' of A containing z_0 such that at every singular point z' of f in $A' - z_0$, the set $C(f, z')$ is $C_B(f, z_0)$.

A theorem of Lohwater's [8, Theorem 8] asserts that if f(z) is meromorphic in D, if the modulus $|f(z)|$ has radial limit 1 almost everywhere on an open subarc of C, and if z_0 is a point of A that is not a limit of zeros or poles of f, then a necessary and sufficient condition for z_0 to be a singularity of f(z) is that every subarc A" of A that contains z_0 also contains a point at which 0 or ∞ is an asymptotic value. Examination of the proof shows that if z_0 is a singularity of f, one can strengthen the conclusion somewhat: If 0 and ∞ are both in $C(f, z_0)$, then both are asymptotic values in A". Applying this result to the arc A' of the last paragraph, we can say that there exist two arcs B and B' terminating at points of A' such that on B, f(z) approaches 0 and on B', f(z) approaches ∞. Since either 0 or ∞ does not belong to the cluster set of any point in $A' - z_0$, one of the arcs B and B' terminates at z_0 and has the property that the corresponding asymptotic value is not in $C_B(f, z_0)$. This establishes the hypotheses of Theorem 2, and our conclusion follows.

I suspect that the following is true.

CONJECTURE. *Let* f(z) *be a bounded analytic function in* D, *and suppose that the modulus of* f *has radial limit* 1 *almost everywhere on an open arc* A *of* C. *If* z_0 *is a singularity of* f *in* A, *then either* z_0 *has property* P_n *for every integer* $n \geq 2$, *or for every arc* B *terminating at* z_0 *the set* $C(f, B, z_0)$ *contains the entire circle* C.

In [2], the authors give an example of a Blaschke product f(z) such that the point $w = 1$ is in the cluster set of every arc terminating at the sole singularity $z = 1$, so that an isolated singularity need not have property P_3. The following example goes somewhat further: it shows that the set common to all arc-cluster sets of arcs terminating at $z = 1$ can vary from the closed disk down to quite thin continua. However, the example does not have a radial limit of modulus 1 almost everywhere.

EXAMPLE 1. *Let* $D_1, D_2, D_3, \cdots$ *be a sequence of simply connected domains lying with their boundaries in the open unit disk* D, *such that no two have a point or a boundary point in common, and such that* $\lim \operatorname{diam} D_n = 0$. *Let* K *denote the set* $(D \cup C) - \bigcup_j D_j$. *Then there exists a function* f(z), *analytic and bounded by* 1 *in* D, *and continuous in* $D \cup C$ *except at the point* $z = 1$, *such that*

(a) $C(f, 1) = D \cup C$,

(b) $C_B(f, 1) = K$,

(c) *if* A *is an arc terminating at* 1, *then* $C(f, A, 1)$ *contains* K,

(d) *there exist uncountably many arcs* A, *terminating at* 1, *for which* $C(f, A, 1) = K$.

Proof. Suppose, for initial simplicity, that $z = 0$ is not in any set $\mathrm{Cl}\, D_k$. Let

$$\phi(z) = \exp\left(\frac{z+1}{z-1}\right).$$

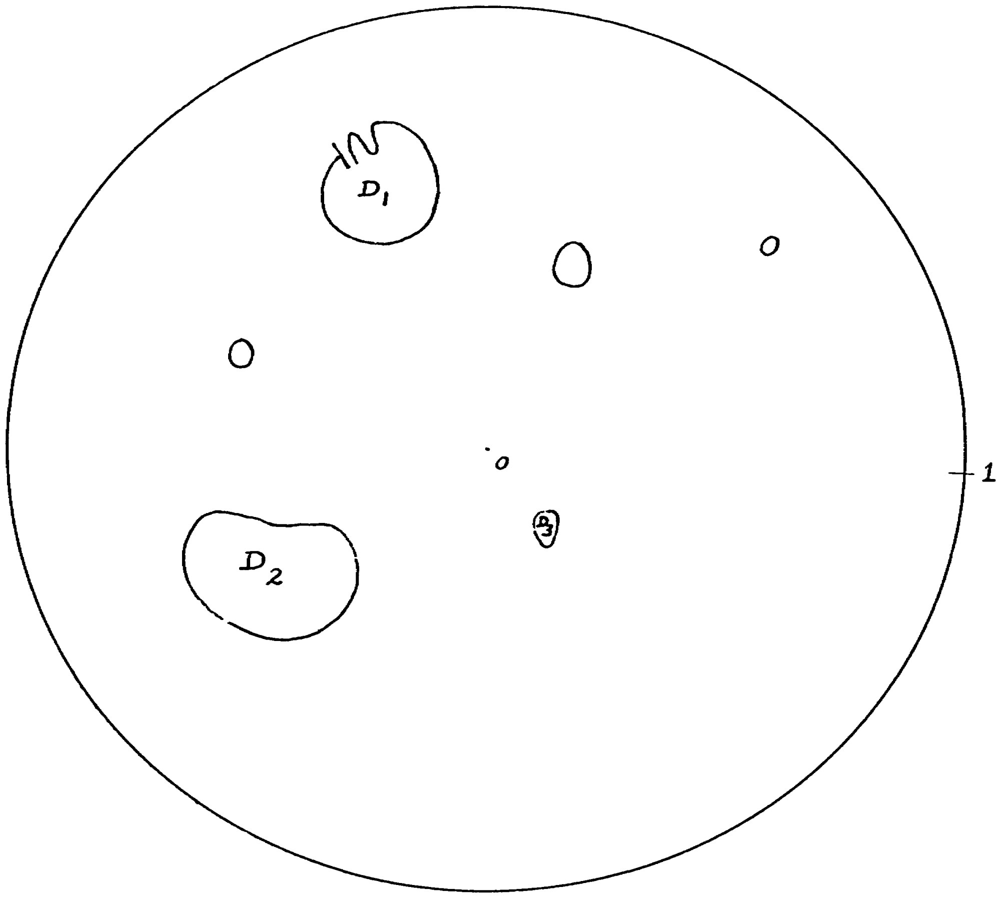

Fig. 1a.

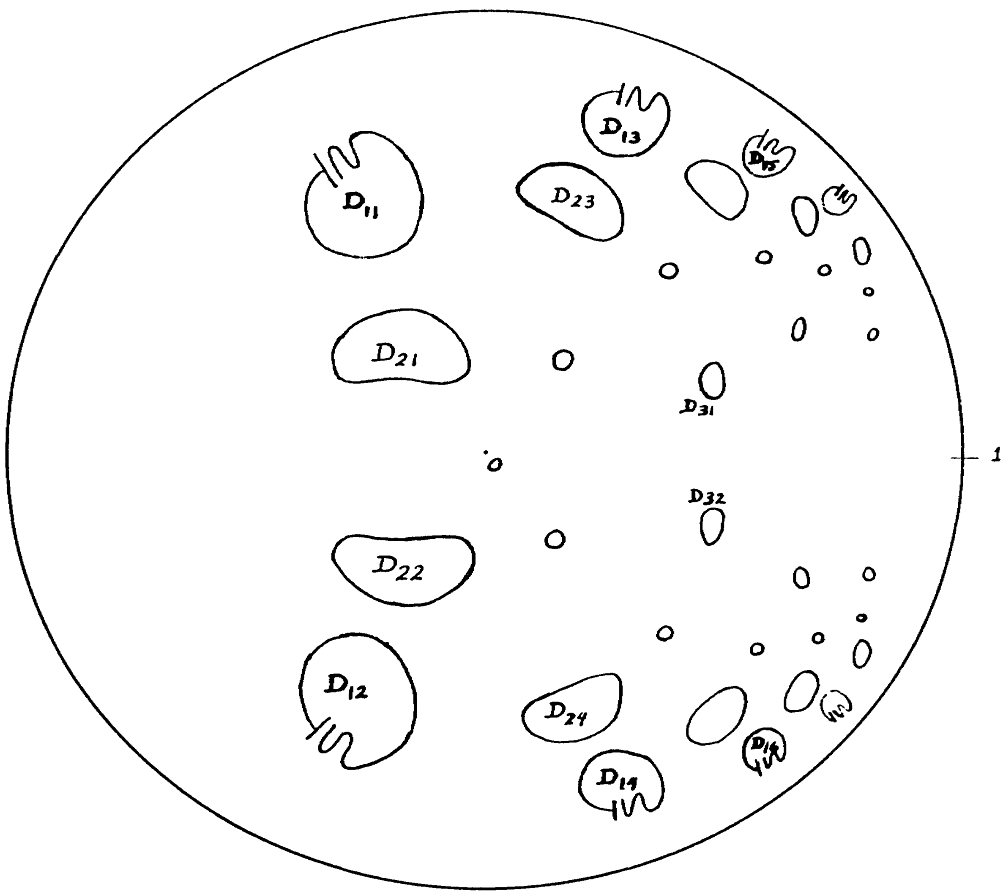

Fig. 1b.

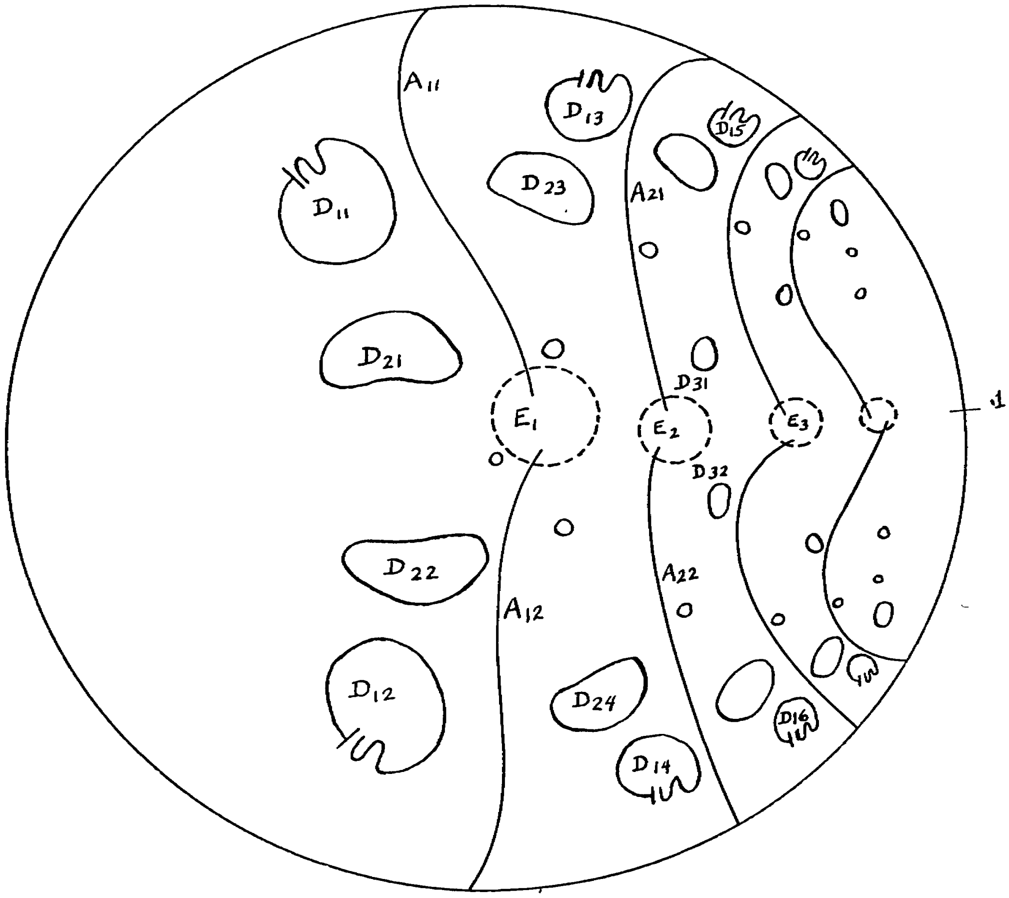

Fig. 2.

Then for each $k = 1, 2, 3, \cdots$, $\phi^{-1}(D_k)$ is the union of an infinite set of simply connected domains D_{kj} ($j = 1, 2, 3, \cdots$) in D; each of the sequences $\{D_{kj}\}_{j=1}^{\infty}$ converges to $z = 1$; and no two domains of the family $\{D_{kj}\}$ have intersecting closures. (See Figure 1.) It follows that the set of all domains D_{kj} ($k, j = 1, 2, 3, \cdots$) satisfies all the conditions imposed on the original domains. Let K' denote the continuum $(C \cup D) - \bigcup_{kj} D_{kj}$.

The range of ϕ at 1, $R(\phi, 1)$, contains all of D except $z = 0$. The set $K \cap (D - 0)$ contains a countable dense subset S. Since $D - 0$ is contained in $R(\phi, 1)$, we can find a sequence $x_1, x_2, x_3, \cdots$ of points of $K \cap D$ converging to $z = 1$ and such that $\bigcup \phi(x_j)$ is S. There exists a sequence $E_1, E_2, E_3, \cdots$ of open disks in D such that for each $j = 1, 2, 3, \cdots$, the point x_j is a point of E_j, and such that if $\{y_j\}$ is any other sequence of points with y_j in E_j ($j = 1, 2, 3, \cdots$), then $\{\phi(y_j)\}$ is also dense in K. (The sets E_j may very possibly overlap some of the sets D_{kj}. Indeed, they all must do so in those cases in which K has no interior points.)

There exists a monotone map m of K' onto Cl D such that the inverse of a point of Cl D consists either of a single point or of the boundary of a domain D_{kj}. This is a consequence of a well-known theorem of R. L. Moore on upper-semicontinuous collections of plane continua [11, p. 171]. The set T of points y in Cl D such that $m^{-1}(y)$ is non-degenerate is countable, and hence each pair of points of Cl D - T can be joined by an arc A in Cl D - T. Since m is one-to-one on the compact set $m^{-1}(A)$, this set is also an arc. It follows that the set

$$K' - \bigcup_{kj} \text{Bdry } D_{kj}$$

is arcwise connected. Using this fact, we can construct in K' a sequence of arcs A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}, $\cdots$ such that

(1) for each n and j ($n = 1, 2, 3, \cdots$, $j = 1, 2$), the arc A_{nj} lies in $K' \cap D$ except for one endpoint, $z_{nj} = \exp i\theta_{nj}$;

(2) the sequence $\{\theta_{n1}\}$ of numbers is strictly decreasing and the sequence $\{\theta_{n2}\}$ is strictly increasing, and each converges to 0;

(3) each arc A_{nj} joins z_{nj} to a point of E_n;

(4) no two arcs A_{nj} intersect;

(5) $\lim A_{nj} = 1$; and

(6) no set $\mathrm{Cl}\, D_{nk}$ intersects any arc A_{nj}. (See Figure 2.)

Let $L = C \cup \bigcup_{nj} A_{nj}$. Then L is a Peano continuum that does not separate D. If A is any arc in $D \cup \{1\}$ that has $z = 1$ as one endpoint, but that has no other point in common with L, then A must intersect all but a finite number of the sets E_n. By the definition of $\{E_n\}$, D(f, A, 1) contains K. The property (6) permits us to construct many arcs in K' - L ending at $z = 1$, and for these, the arc cluster set is exactly K.

The set D - L is simply connected, so that there exists a conformal map $\psi(z)$ of D onto D - L. Since L is a Peano continuum, ψ can be extended to be continuous on $D \cup C$. There is no loss in assuming that $\psi(1) = 1$. Note that if B is an arc in $D \cup \{1\}$ approaching 1, then $\psi(B)$ intersects all but a finite number of the sets E_n.

Now let $f(z) = \phi(\psi(z))$. Then f(z) satisfies all the desired conditions. If $z = 0$ does lie in a set $\mathrm{Cl}\, D_k$, there exists a linear map that leaves the circle C fixed and sends $z = 0$ into a point of K. The composition of this map and f is the desired map.

That one can have a bounded analytic function on the disk with a point of discontinuity on the circle such that every two arc cluster sets at that point intersect is of course not new. One needs only consider a conformal map from the disk onto a simply connected domain not all of whose prime ends are of the first kind.

I now show that part of the phenomenon of Example 1 is not due to the point 1 being an isolated discontinuity of the function f(z) in that example. The example also gives a partial answer to Question 4 in [2]. This question concerns a function having radial limits of modulus 1 almost everywhere on the circle, and it asks whether the nonisolated singularities of f on the circle have property P_3 or are at least limits of points with property P_3.

EXAMPLE 2. *Let* K *be a set satisfying the conditions on the set* K *in Example* 1. *Then there exists a function* f(z), *analytic and bounded by* 1 *in* D, *and continuous in* $D \cup C$ *except at the points of a Cantor set* T, *such that for each point* t *in* T

(a) C(f, t) *is contained in* $D \cup C$,

(b) $C_B(f, t) = K$,

(c) *if* A *is an arc terminating at* t, *then* C(f, A, t) *contains* K,

(d) *there exist uncountably many arcs* A *terminating at* t *for which* C(f, A, t) = K.

Proof. For each pair of integers n and k ($n \geq 2$ and k odd) such that $0 < k/2^n < 1$, construct a line segment I_{nk} of length $1/2^n$, with one endpoint at

$z = 0$, and making an angle $k\pi/2^n$ with the positive real axis. Let T" denote the union of all the intervals I_{nk} thus defined.

If f(z) denotes the map constructed in Example 1, the components of the set $f^{-1}(D - K)$ are simply connected open sets with pairwise disjoint closures H_j. Again using Moore's theorem, we obtain a monotone map m of Cl D onto itself such that the inverse of a point in Cl D is either a point in $f^{-1}(K) - \bigcup_j H_j$ or is a set H_j, and such that $m(1) = 1$. The sequence of points $m(H_1)$, $m(H_2)$, $\cdots$ is countable, and thus is easily avoided, so that we can construct a set T' in $D \cup \{1\}$ that is homeomorphic to the set T", has its "interesting point" at $z = 1$, and contains none of the points $m(H_j)$. Let T denote $m^{-1}(T')$. Then T is homeomorphic to T', and $f(T - \{1\})$ lies entirely in K. Let D' denote D - T. Then D' is simply connected, so that there exists a conformal map $\xi(z)$ of D onto D'. Since $C \cup T$ is a Peano continuum, ξ can be extended to be continuous on all of Cl D. The inverse of $z = 1$ under ξ is a Cantor set on C. The desired map is $f(\xi(z))$.

We cannot replace the words "is contained in $D \cup C$" in part (a) by "is equal to $D \cup C$," for Collingwood [5] has recently shown that the set of points on C where the boundary cluster set is not equal to the cluster set is countable, even for purely arbitrary complex functions defined in D.

REFERENCES

1. F. Bagemihl, *Curvilinear cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 379-382.

2. F. Bagemihl, G. Piranian, and G. S. Young, *Intersections of cluster sets*, Bul. Inst. Politehn. Iaşi (N.S.) 5 (1959), 29-34.

3. F. Bagemihl and W. Seidel, *Functions of bounded characteristic with prescribed ambiguous points*, Michigan Math. J. 3 (1955-56), 77-81.

4. C. Carathéodory, *Zum Schwarzschen Spiegelungsprinzip*, Comment. Math. Helv. 19 (1946-47), 263-278.

5. E. F. Collingwood, *Cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 1236-1242.

6. J. L. Doob, *On a theorem of Gross and Iversen*, Ann. of Math. (2) 33 (1932), 753-757.

7. W. Gross, *Zum Verhalten der konformen Abbildung am Rande*, Math. Z. 3 (1919), 43-64.

8. A. J. Lohwater, *The reflection principle and the distribution of values of functions defined in a circle*, Ann. Acad. Sci. Fenn., Series A. I. no. 229 (1956), 1-18.

9. A. J. Lohwater and G. Piranian, *The sets of ambiguous points of functions of bounded characteristic*, Michigan Math. J. 4 (1957), 23-24.

10. W. Seidel, *On the cluster values of analytic functions*, Trans. Amer. Math. Soc. 34 (1933), 1-21.

11. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28, 1942.

12. G. S. Young, *A generalization of Bagemihl's theorem on ambiguous points*, Michigan Math. J. 5 (1958), 223-227.

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