MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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1. INTRODUCTION

Kaplan [4] has called a function f(z), analytic in $|z| < 1$, close-to-convex in $|z| < 1$ if there exists an analytic function $\phi(z)$, convex and schlicht in $|z| < 1$, such that

$$(1.1) \qquad \Re \frac{f'(z)}{\phi'(z)} > 0 \qquad (|z| < 1).$$

Since $F(z) = z\phi'(z)$ is star-like with respect to the origin and schlicht in $|z| < 1$, (1.1) may be written in the alternative form

$$(1.1') \qquad \Re \frac{zf'(z)}{F(z)} > 0 \qquad (|z| < 1).$$

Kaplan has shown [4] that every close-to-convex function f(z) is schlicht in $|z| < 1$, and that, under the hypothesis that f'(z) does not vanish in $|z| < 1$, the condition (1.1) is equivalent to the alternative condition

$$(1.2) \qquad \int_{\theta_1}^{\theta_2} \Re \left(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\pi$$

for $\theta_1 < \theta_2$, $0 < r < 1$. The geometric interpretation of (1.2) is that $w = f(z)$ maps each circle $|z| = r < 1$ onto a simple closed curve whose tangent rotates, as θ increases, in such a way that it never turns back on itself sufficiently in the clockwise direction to reverse its direction completely.

In this paper we extend the concept of close-to-convex functions to meromorphic functions

$$(1.3) \qquad f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots$$

regular in $0 < |z| < 1$ and with a simple pole at the origin. We say that f(z), when given by (1.3), is close-to-convex in the punctured circle $0 < |z| < 1$ relative to F(z) if there exists a meromorphic, star-like, schlicht function F(z) in $|z| < 1$ with a simple pole at the origin, given by

$$(1.4) \qquad F(z) = \frac{b_{-1}}{z} + b_0 + b_1 z + \cdots + b_n z^n + \cdots \qquad (b_{-1} \neq 0),$$

such that

Received April 7, 1961.

$$\Re \frac{zf'(z)}{F(z)} > 0 \quad \text{for } |z| < 1. \tag{1.5}$$

It is to be noticed that in the meromorphic case (1.5) does not imply that f(z) is schlicht, as (1.1') implies when f(z) and F(z) are regular in $|z| < 1$. For example if $F(z) = -z^{-1}$ and

$$-z^2(1 - z^2)f'(z) = 1 + z^2, \quad f(z) = z^{-1} - 2z - \frac{2}{3}z^3 + \cdots \qquad (0 < |z| < 1),$$

then f(z) is not schlicht in $0 < |z| < 1$, since $|a_1| > 1$. Nevertheless, when f(z) is meromorphic and satisfies (1.5), then $w = f(z)$ maps each circle $|z| = r$ $(0 < r < 1)$ onto a smooth curve which, although it may intersect itself, may also have (as in the regular case) "hairpin" turns, provided a complete reversal of the direction of the tangent does not occur. By a modification of Kaplan's argument [4], we show that in the meromorphic case (1.5) is equivalent to

$$\int_{\theta_1}^{\theta_2} \Re\left(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta < \pi \tag{1.6}$$

for $\theta_1 < \theta_2$, $0 < r < 1$, if f(z) is analytic with a nonvanishing derivative in $0 < |z| < 1$ and with a simple pole at $z = 0$.

Considerable interest has been shown for the coefficient problem for meromorphic schlicht functions

$$f(z) = \frac{1}{z} + \sum_{1}^{\infty} a_n z^n \qquad (0 < |z| < 1). \tag{1.7}$$

It is known that

$$|a_n| \leq \frac{2}{n+1} \qquad \text{for } n = 1, 2, \tag{1.8}$$

but that $|a_3|$ may be as large as $1/2 + e^{-6}$ [3]. Recently, Clunie [1] has shown that (1.8) is true for all n whenever f(z) is univalently star-like in $0 < |z| < 1$. This was also shown earlier for $1 \leq n \leq 6$ by Nehari and Netanyahu [5]; see also H. Nishimiya [6] for $1 \leq n \leq 4$. On the other hand Clunie [2] has recently shown that the order $O(n^{-1})$ is not the correct one for a_n in the general case. In this paper, we show that if f(z) is schlicht and close-to-convex in $0 < |z| < 1$ with a simple pole at $z = 0$, then the order $O(n^{-1})$ is the correct one for a_n. More precisely, we prove the following three theorems.

THEOREM 1. *Let*

$$f(z) = \frac{1}{z} + \sum_{0}^{\infty} a_n z^n \tag{1.9}$$

be regular, schlicht and close-to-convex in $0 < |z| < 1$, *relative to the meromorphic, star-like and schlicht function*

(1.10) $$F(z) = \frac{b_{-1}}{z} + \sum_0^\infty b_n z^n \qquad (0 < |z| < 1, \ |b_{-1}| = 1);$$

then

(1.11) $$n|a_n| + |b_n| \leq c \qquad (n = 1, 2, \cdots),$$

where c *is a real constant,* $2 \leq c \leq 2\sqrt{2}$.

If f(z) in (1.9) is star-like, we may take $F(z) = -f(z)$ in (1.10), so that $|b_n| = |a_n|$. In this case (1.11) reduces (even when $a_0 \neq 0$) to

(1.12) $$|a_n| \leq \frac{c}{n+1} \qquad \text{with } c = 2;$$

this was shown by Clunie [1] (who assumed $a_0 = 0$, although his method of proof still holds for $a_0 \neq 0$). We were unable to show that $c = 2$ for all close-to-convex, schlicht functions (1.9).

If f(z) is given by (1.3) and if for each r $(0 < r < 1)$ the image curve Γ_r, corresponding to $|z| = r$ through the mapping $w = f(z)$, has the property that each straight line parallel to some fixed direction cuts Γ_r in at most two points, we say that f(z) is convex in that direction. If f(z) is meromorphic and convex in one direction, it need not be schlicht in $0 < |z| < 1$. An example is given in (3.20).

THEOREM 2. *Let*

(1.13) $$f(z) = \frac{1}{z} + \sum_0^\infty a_n z^n$$

be regular and convex in the direction of the imaginary axis, in $0 < |z| < 1$, *and real on the real axis. Then*

(1.14) $$-1 \leq a_1 \leq 3,$$

(1.15) $$|a_n| \leq \frac{2(1 + a_1)^{1/2}}{n} \leq \frac{4}{n} \qquad (n > 1),$$

and these inequalities are sharp for n *odd. If* f(z) *is also schlicht, then*

(1.16) $$|a_n| \leq 2\sqrt{2}/n, \qquad |a_1| \leq 1.$$

THEOREM 3. *If* f(z), *given by* (1.3), *is an odd function convex in the directions of both coordinate axes, for* $0 < |z| < 1$, *and if the coefficients* a_n *are all real, then* f(z) *is necessarily schlicht in* $0 < |z| < 1$, *and*

(1.17) $$|a_n| \leq \frac{2(1 - |a_1|)^{1/2}}{n} \qquad (n \text{ odd}, > 1).$$

2. CLOSE-TO-CONVEX FUNCTIONS

Let f(z) and F(z) be defined as in (1.3) and (1.4), and let (1.5) hold. Using the notation of Kaplan [4], we let $i \exp[iP(r, \theta)]$ be the unit tangent vector to the image curve of $|z| = r$ $(0 < r < 1)$ through the mapping $w = f(z)$, and we let

$$Q(r, \theta) = \arg F(re^{i\theta}).$$

Because of (1.5) (and with proper choice of the argument) we have

$$|P(r, \theta) - Q(r, \theta)| < \pi/2. \tag{2.1}$$

Since F(z) is meromorphic and star-like, we also have $\partial Q/\partial\theta < 0$. Then, by an argument similar to that used by Kaplan [4, p. 173], we obtain

$$P(r, \theta_1) - P(r, \theta_2) > -\pi \qquad \text{for } \theta_1 < \theta_2, \tag{2.2}$$

and (2.2) is readily seen to be equivalent to

$$\int_{\theta_1}^{\theta_2} \Re\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta < \pi. \tag{2.3}$$

Thus (1.5) now implies (2.3) instead of (1.2).

Conversely, if f(z) is given by (1.3) and $f'(z) \neq 0$ in $0 < |z| < 1$, then (2.3) implies the existence of a star-like, schlicht function F(z) of the form (1.4) such that (1.5) holds, that is to say, f(z) is then close-to-convex (although not necessarily schlicht). Since f(z) is no longer analytic at $z = 0$, the argument of Kaplan [4, pp. 174-176] must be modified, although the general procedure is much the same. We have

$$P(r, \theta + 2\pi) - P(r, \theta) = -2\pi. \tag{2.4}$$

LEMMA. *Let* $t(\theta)$ *be a real function of* θ *for* $-\infty < \theta < \infty$ *such that*

$$t(\theta + 2\pi) - t(\theta) = -2\pi, \tag{2.5}$$

$$t(\theta_1) - t(\theta_2) > -\pi \qquad \text{for } \theta_1 < \theta_2; \tag{2.6}$$

then there exists a real function $s(\theta)$ *which is non-increasing and satisfies the conditions*

$$s(\theta + 2\pi) - s(\theta) = -2\pi, \tag{2.7}$$

$$|s(\theta) - t(\theta)| \leq \pi/2. \tag{2.8}$$

The proof of the lemma is the same as Kaplan's [4], except that now $s(\theta)$ is defined as

$$s(\theta) = \underset{\theta' \leq \theta}{\text{g.l.b.}}\, t(\theta') + \pi/2. \tag{2.9}$$

Take $t(\theta) = P(\rho, \theta)$ $(0 < \rho < 1)$, and denote the corresponding $s(\theta)$ of the lemma by $s(\rho, \theta)$. For $r < \rho$, define

$$q_\rho(r, \theta) = \frac{\rho^2 - r^2}{2\pi}\int_0^{2\pi} \frac{(s(\rho, \alpha) + \alpha)\, d\alpha}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)}, \tag{2.10}$$

so that $q_\rho(r, \theta)$ is harmonic for $r < \rho$. This definition of $q_\rho(r, \theta)$ differs from the one used by Kaplan in the regular case, since $s(\rho, \alpha) - \alpha$ in his definition is replaced in (2.10) by $s(\rho, \alpha) + \alpha$. Define the function

$$Q_\rho(r, \theta) = q_\rho(r, \theta) - \theta \tag{2.11}$$

(replacing $q_\rho(r, \theta) + \theta$ used by Kaplan). Then, since $s(\rho, \alpha + \theta_2) - s(\rho, \alpha + \theta_1)$ has a period 2π, it follows by straightforward computation that for $\theta_1 < \theta_2$

$$Q_\rho(r, \theta_2) - Q_\rho(r, \theta_1) = \frac{\rho^2 - r^2}{2\pi}\int_0^{2\pi} \frac{s(\rho, \alpha + \theta_2) - s(\rho, \alpha + \theta_1)}{\rho^2 + r^2 - 2\rho r \cos\alpha}\, d\alpha < 0. \tag{2.12}$$

We next define $h_\rho(z)$ to be the analytic completion of $q_\rho(r, \theta)$, so that the imaginary part of $h_\rho(z)$ is identical with the harmonic function $q_\rho(r, \theta)$, and such that $\Re h_\rho(0) = 0$. We define $F_\rho(z)$ by the equation

$$F_\rho(z) = \frac{1}{z} \exp[h_\rho(z)] \neq 0, \tag{2.13}$$

and we write

$$F_\rho(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots \qquad (0 < |z| < \rho). \tag{2.14}$$

Then

$$c_{-1} = e^{h_\rho(0)}, \qquad |c_{-1}| = |e^{i\Im h_\rho(0)}| = 1. \tag{2.15}$$

$F_\rho(z)$ is analytic for $0 < |z| < \rho$ and has a simple pole at the origin. We also have

$$\Re \frac{zF'_\rho(z)}{F_\rho(z)} = \Re[zh'_\rho(z) - 1] = \frac{\partial q_\rho(r, \theta)}{\partial\theta} - 1 = \frac{\partial}{\partial\theta} Q_\rho(r, \theta) < 0. \tag{2.16}$$

Hence $F_\rho(z)$ is a star-like, schlicht function for $0 < |z| < \rho$. Following the argument of Kaplan [4, p. 176], we choose a sequence $\rho_n \to 1$ so that $F_{\rho_n}(z) \to F(z)$ uniformly in every closed domain within the unit circle. $F(z)$ is then schlicht and star-like in $0 < |z| < 1$, and

$$\Re\left(\frac{zf'(z)}{F(z)}\right) > 0 \qquad (0 < |z| < 1), \tag{2.17}$$

as in the regular case.

3. BOUNDS ON THE COEFFICIENTS

Let f(z) and F(z), defined by (1.9) and (1.10), respectively, satisfy the conditions of Theorem 1. We may assume that $|b_{-1}| = 1$, $b_{-1} = e^{i\alpha}$, and that (1.5) holds. Then there exists a bounded, regular function $\omega(z)$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $|z| < 1$, such that

$$-\sec\alpha \frac{zf'(z)}{F(z)} + i\tan\alpha = \frac{1+\omega(z)}{1-\omega(z)}, \tag{3.1}$$

$$\omega'(0) = -\frac{b_0}{2} e^{-2i\alpha} \sec\alpha, \qquad -\cos\alpha > 0. \tag{3.2}$$

Moreover,

$$[z^2 f'(z) - e^{i\alpha} zF(z)]\omega(z) = [z^2 f'(z) + e^{-i\alpha} zF(z)], \tag{3.3}$$

$$\left[-2e^{i\alpha}\cos\alpha + \sum_0^\infty (ka_k - e^{i\alpha} b_k) z^{k+1}\right]\omega(z) = \sum_0^\infty (ka_k + e^{-i\alpha} b_k) z^{k+1}, \tag{3.4}$$

$$\begin{aligned}&\left[-2e^{i\alpha}\cos\alpha + \sum_0^{n-1} (ka_k - e^{i\alpha} b_k) z^{k+1}\right]\omega(z) \\ &= \sum_0^n (ka_k + e^{-i\alpha} b_k) z^{k+1} + \sum_{n+2}^\infty c_k z^k,\end{aligned} \tag{3.5}$$

where $\Sigma_{n+2}^\infty c_k z^k$ converges in $|z| < 1$. Let $z = re^{i\theta}$ $(r < 1)$. An integration gives

$$\begin{aligned}&4\cos^2\alpha + \sum_0^{n-1} |ka_k - e^{i\alpha} b_k|^2 r^{2k+2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|-2e^{i\alpha}\cos\alpha + \sum_0^{n-1} (ka_k - e^{i\alpha} b_k) z^{k+1}\right|^2 d\theta \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \left|-2e^{i\alpha}\cos\alpha + \sum_0^{n-1} (ka_k - e^{+i\alpha} b_k) z^{k+1}\right|^2 \cdot |\omega(z)|^2\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_0^n (ka_k + e^{-i\alpha} b_k) z^{k+1} + \sum_{n+2}^\infty c_k z^k\right|^2 d\theta \\ &\geq \sum_0^n |ka_k + e^{-i\alpha} b_k|^2 r^{2k+2} \qquad (r < 1),\end{aligned} \tag{3.6}$$

$$4\cos^2\alpha + \sum_0^{n-1} |ka_k - e^{i\alpha} b_k|^2 \geq \sum_0^n |ka_k + e^{-i\alpha} b_k|^2 , \tag{3.7}$$

$$\begin{aligned} |na_n + e^{-i\alpha} b_n|^2 &\leq 4\cos^2\alpha - \sum_0^{n-1} \{|ka_k + e^{-i\alpha} b_k|^2 - |ka_k - e^{i\alpha} b_k|^2\} \\ &= 4\cos^2\alpha - 4\cos\alpha\, \Re \sum_0^{n-1} ka_k \bar{b}_k , \end{aligned} \tag{3.8}$$

$$n^2 |a_n|^2 + |b_n|^2 \leq 4\cos^2\alpha - 4\cos\alpha\, \Re \sum_0^{n-1} ka_k \bar{b}_k - 2\Re(na_n \bar{b}_n e^{i\alpha}) , \tag{3.9}$$

$$\begin{aligned} (n|a_n| + |b_n|)^2 &\leq 4\cos^2\alpha - 4\cos\alpha \sum_1^{n-1} k|a_k b_k| + 4n|a_n b_n| \\ &\leq 4\left[1 + \sum_1^n k|a_k b_k|\right]. \end{aligned} \tag{3.10}$$

Since $F(z)$ is univalent in $0 < |z| < 1$, it follows by the area theorem that

$$\sum_1^\infty k|b_k|^2 \leq 1 \qquad (|b_{-1}| = 1) . \tag{3.11}$$

If we assume that $f(z)$ is schlicht in $0 < |z| < 1$, we also have, by the area theorem,

$$\sum_1^\infty k|a_k|^2 \leq 1 . \tag{3.12}$$

An application of the Schwarz inequality yields

$$\sum_1^n k|a_k b_k| \leq \left(\sum_1^n k|a_k|^2\right)^{1/2} \left(\sum_1^n k|b_k|^2\right)^{1/2} \leq 1 . \tag{3.13}$$

By (3.10) and (3.13), we may now write

$$n|a_n| + |b_n| \leq 2\sqrt{2} \qquad (n = 1, 2, \cdots) . \tag{3.14}$$

It should be noticed that we may replace the condition that $f(z)$ be univalent in $0 < |z| < 1$ by the weaker condition (3.12), in order to obtain the bounds (3.14). This completes the proof of Theorem 1.

If $f(z)$ satisfies the conditions of Theorem 2, we then have, on $|z| = r$ $(0 < r < 1)$,

$$\Im\, zf'(z) = -\frac{\partial}{\partial\theta}\Re f(re^{i\theta}) \begin{cases} > 0 & (0 < \theta < \pi), \\ < 0 & (\pi < \theta < 2\pi). \end{cases} \tag{3.15}$$

Thus

$$\Re\left(\frac{-z^2 f'(z)}{1 - z^2}\right) > 0 \qquad (|z| < 1). \tag{3.16}$$

It follows that f(z) is close-to-convex in $0 < |z| < 1$, relative to the star-like function

$$F(z) = -\frac{1}{z} + z. \tag{3.17}$$

Since $b_{-1} = -1$, $\alpha = \pi$, $b_1 = 1$ and $b_n = 0$ for $n > 1$, we see from (3.8) with $n = 1$ that

$$|a_1 - 1|^2 \leq 4, \qquad -1 \leq a_1 \leq 3; \tag{3.18}$$

and, for $n > 1$,

$$n^2 |a_n|^2 \leq 4\,(1 + a_1) \leq 16, \qquad |a_n| \leq 4/n. \tag{3.19}$$

If f(z) is also schlicht, then $|a_1| \leq 1$ and $|a_n| \leq 2\sqrt{2}/n$ follows. The inequalities (3.19) are sharp for all odd values of $n > 1$, as the following example shows. The function

$$f(z) = \frac{1}{z} + 3z + 4\sum_{1}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} \tag{3.20}$$

satisfies the equation

$$\frac{-z^2 f'(z)}{1 - z^2} = \frac{1 - z^2}{1 + z^2}. \tag{3.21}$$

Therefore it satisfies (3.16). It is not schlicht, since $|a_1| > 1$. This completes the proof of Theorem 2.

In Theorem 3, f(z) is odd and convex in the directions of the axes of reference, in addition to being real on the real axis. Consequently both f(z) and if(iz) satisfy the conditions of Theorem 2. Thus (3.19) holds with a_1 replaced by $-|a_1|$, and hence reduces to

$$n^2 |a_n|^2 \leq 4(1 - |a_1|) \qquad (n \text{ odd}, > 1). \tag{3.22}$$

It is easily seen that f(z) has to be schlicht in $0 < |z| < 1$. This completes the proof of Theorem 3.

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